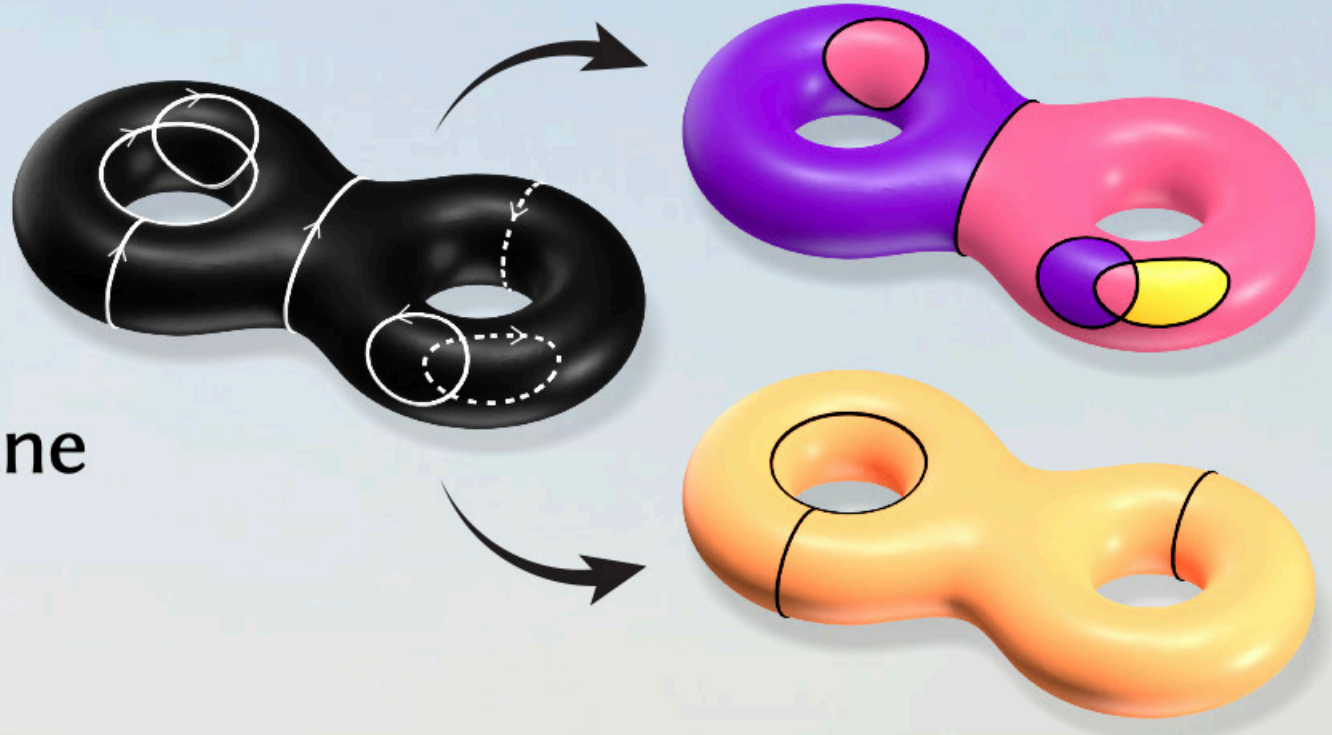
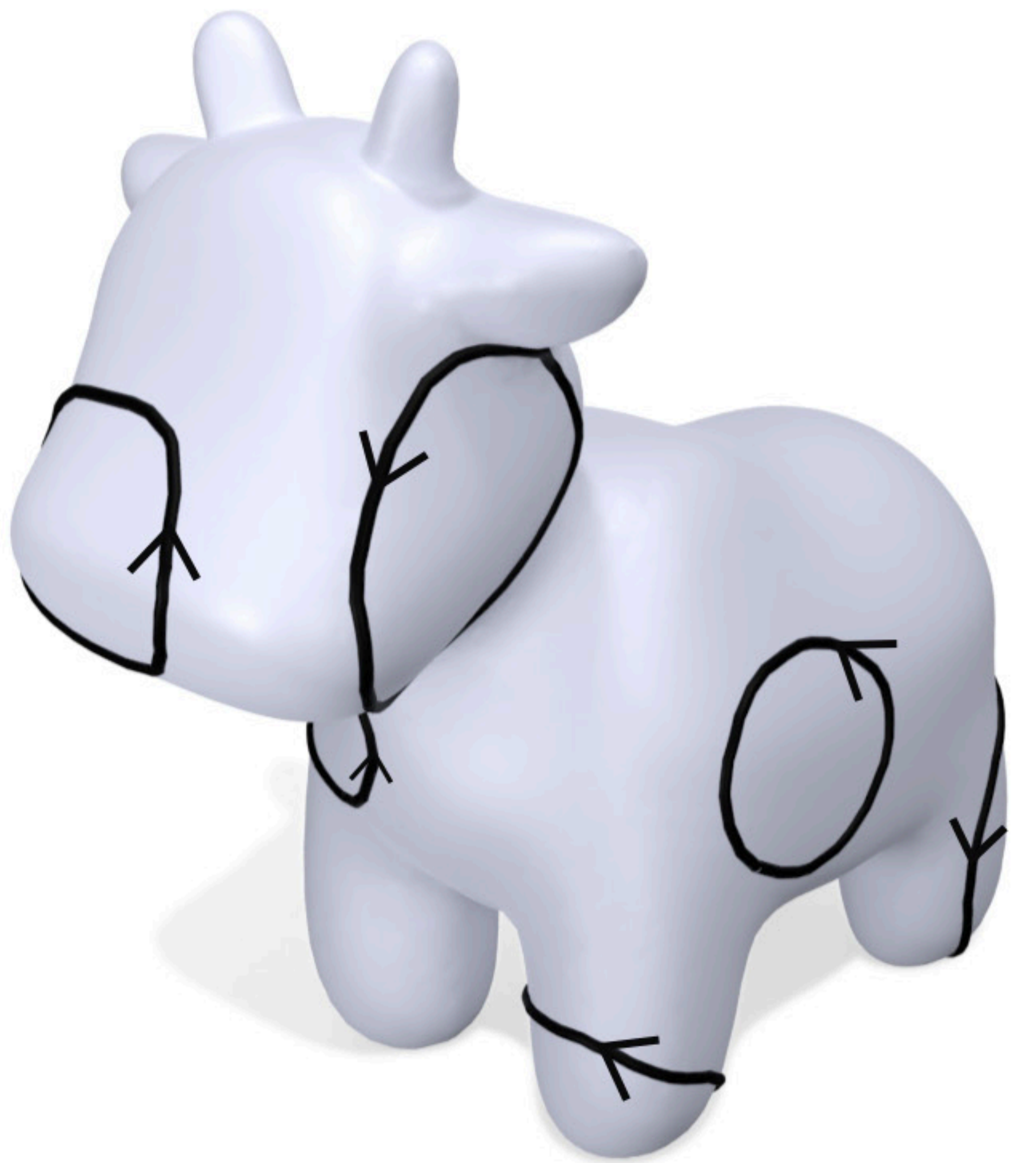
Winding Numbers on Discrete Surfaces

Nicole Feng, Mark Gillespie, Keenan Crane
Carnegie Mellon University



[date]

Problem



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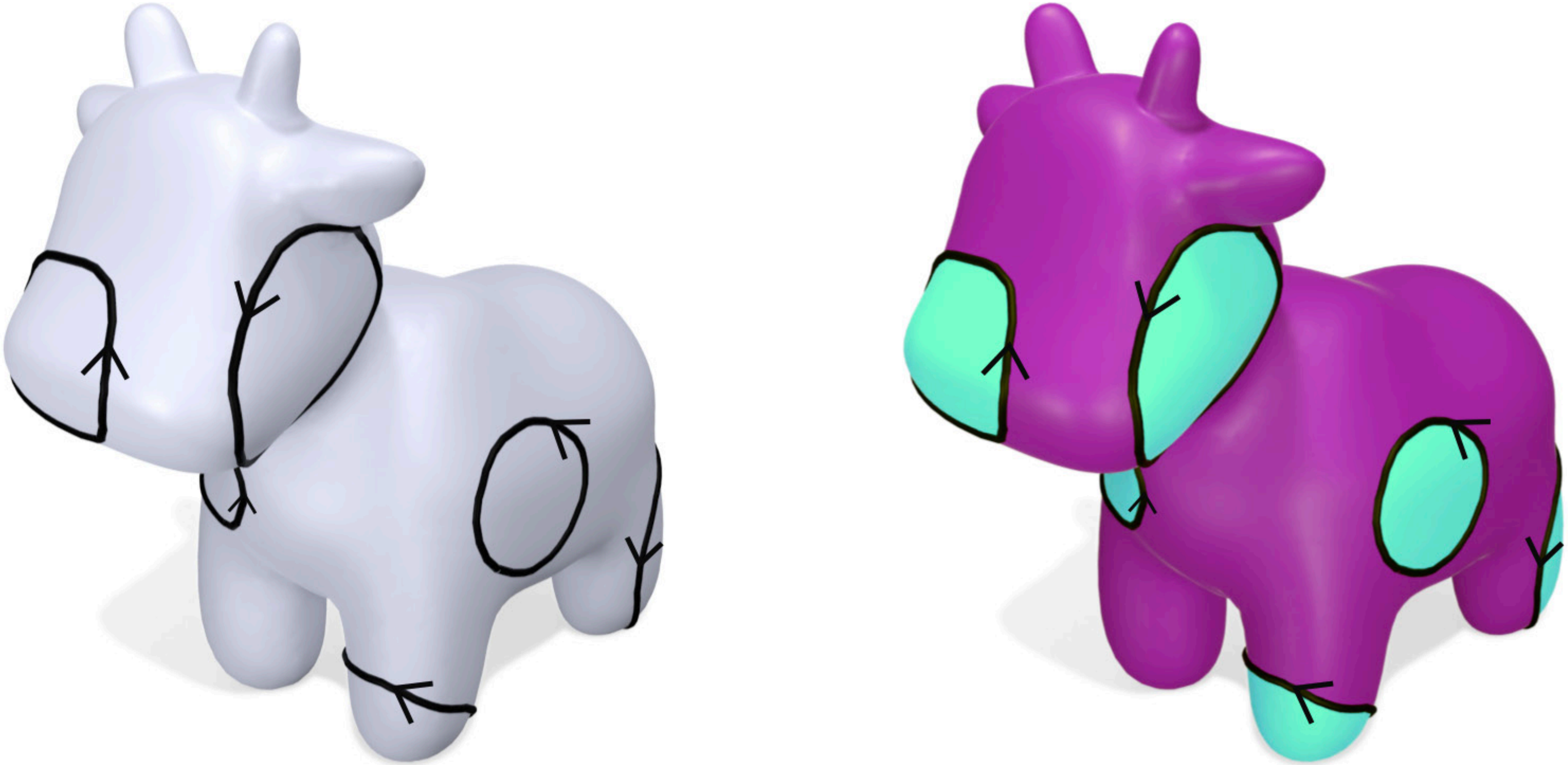

Not all curves bound regions!

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Fence bounds a region:

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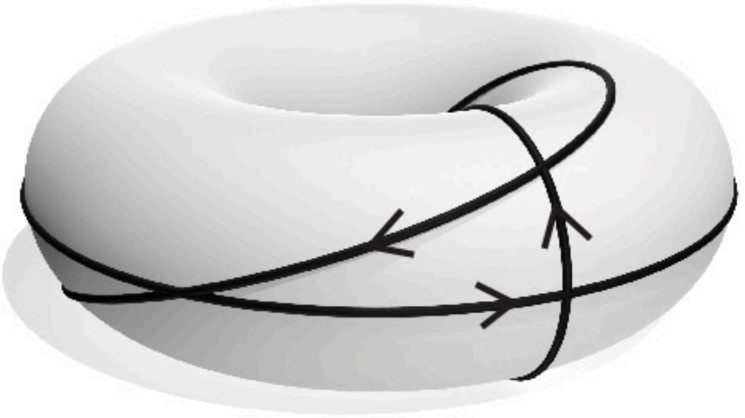
Fence bounds a region:

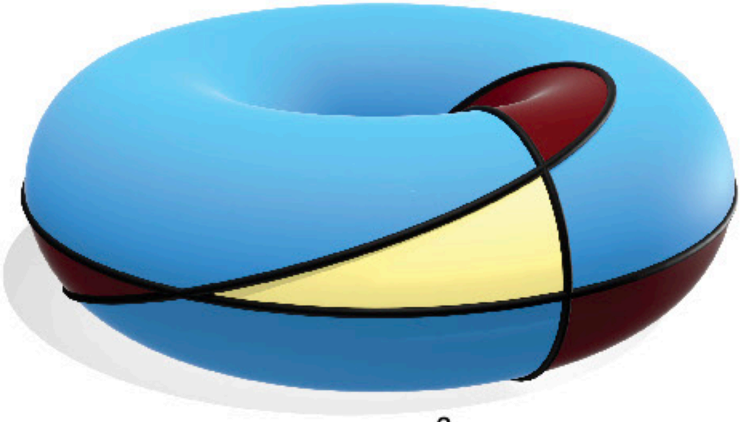
Fence doesn't bound a region:

Not all curves bound regions!

Fence bounds a region:

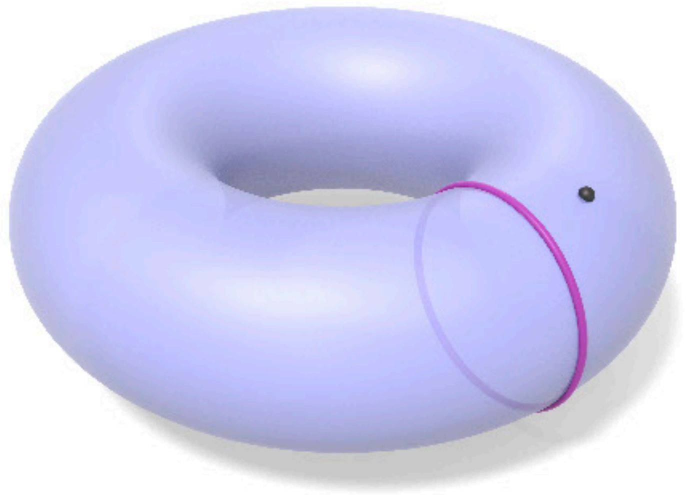
Fence doesn’t bound a region:





Not as simple as classifying loops

Is the point inside or outside the curve?



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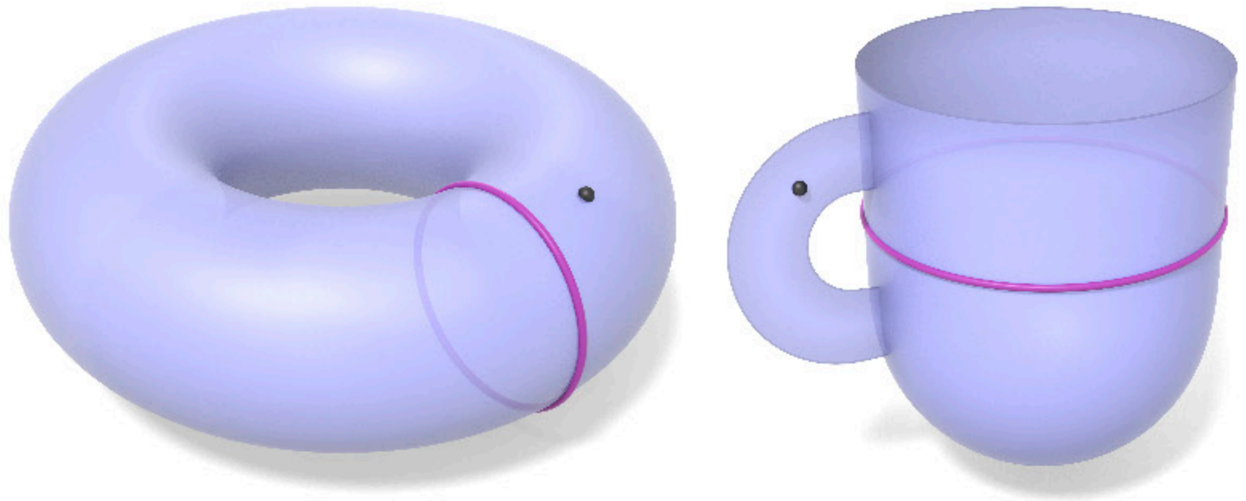
non-manifold



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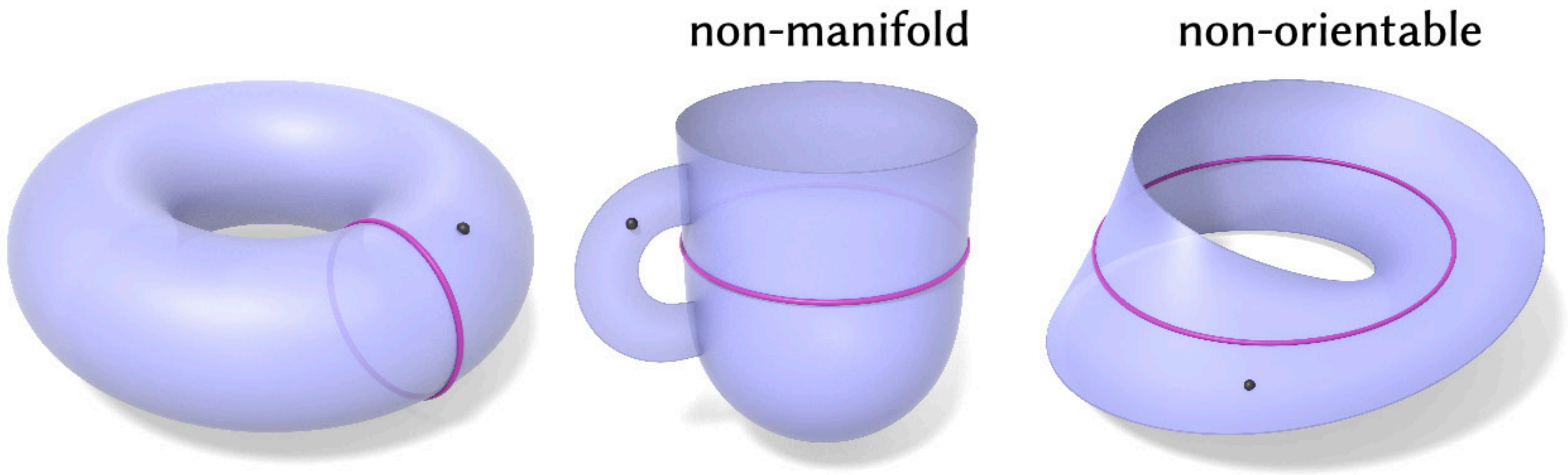

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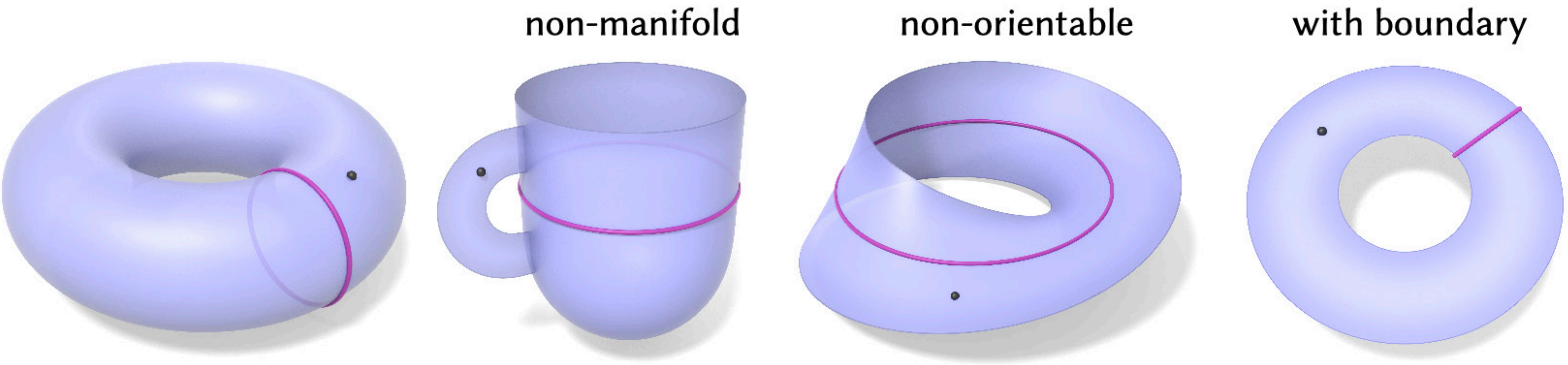

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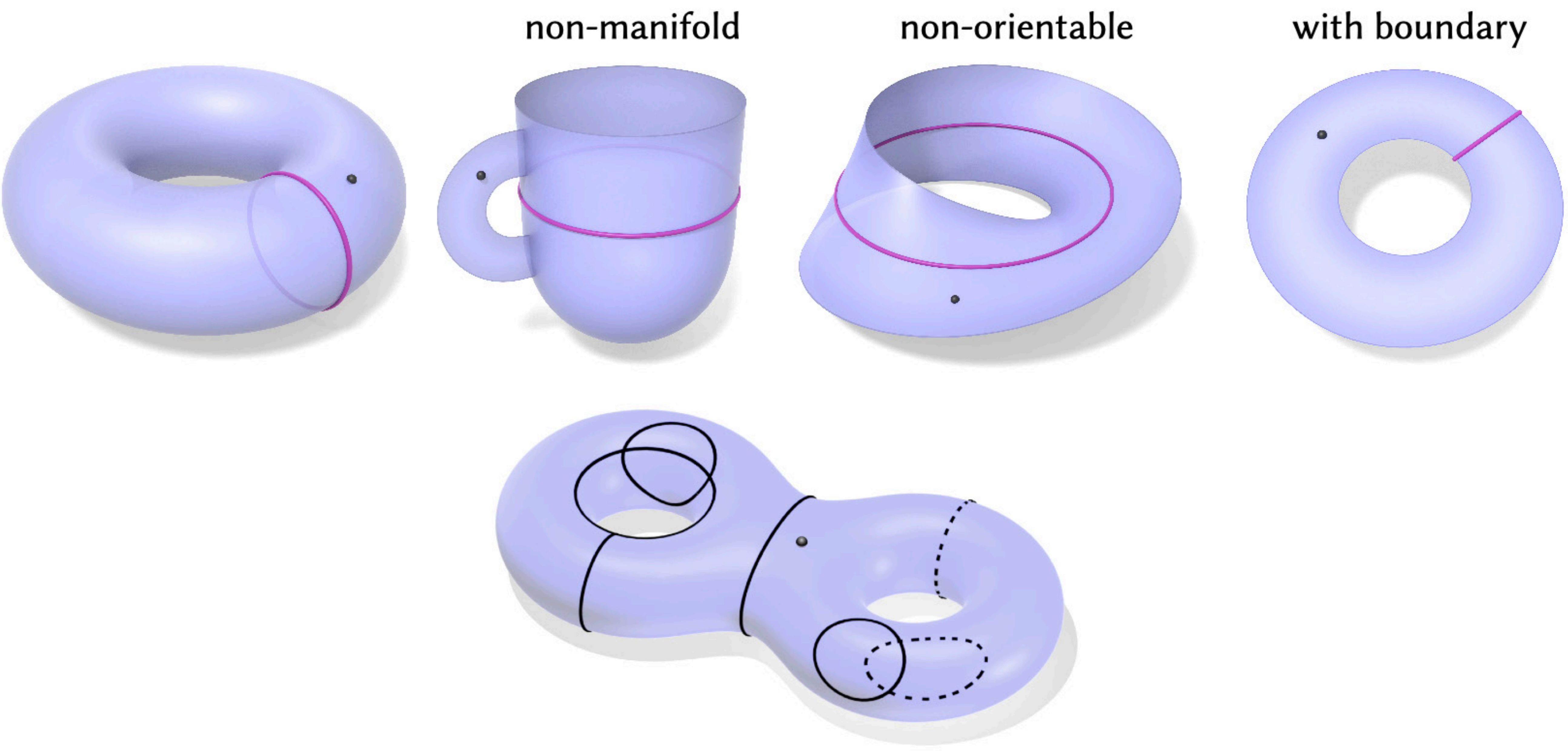
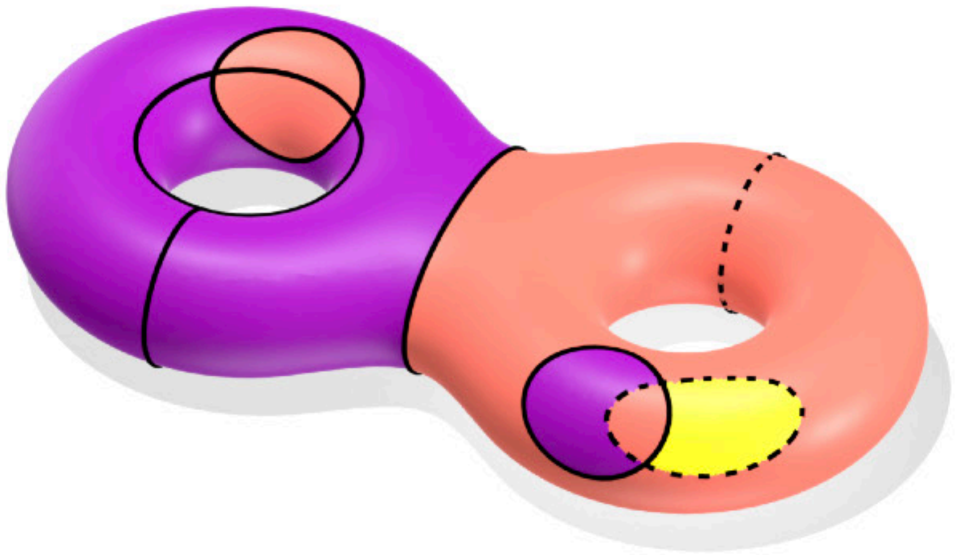
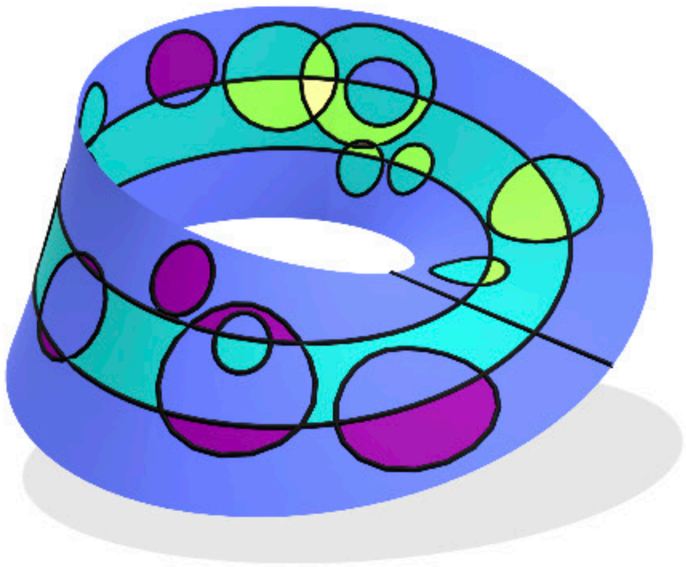
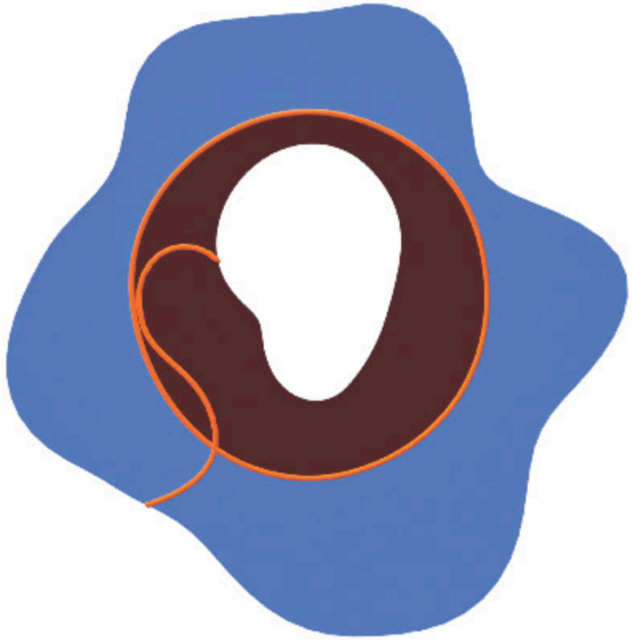


Surface Winding Numbers (SWN)







Winding number (solid angle)

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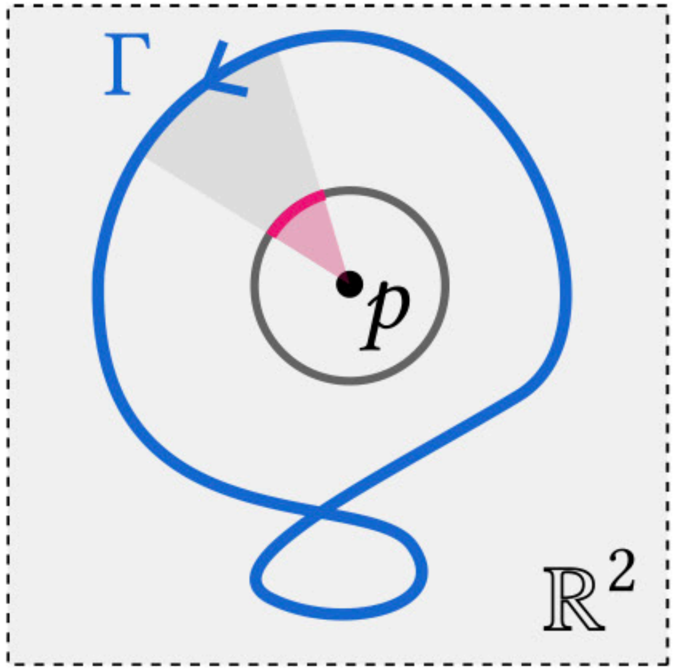

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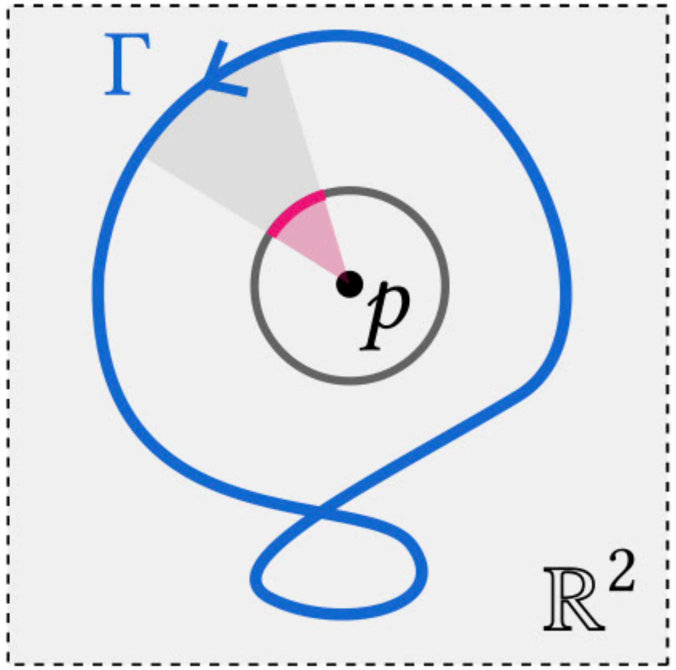

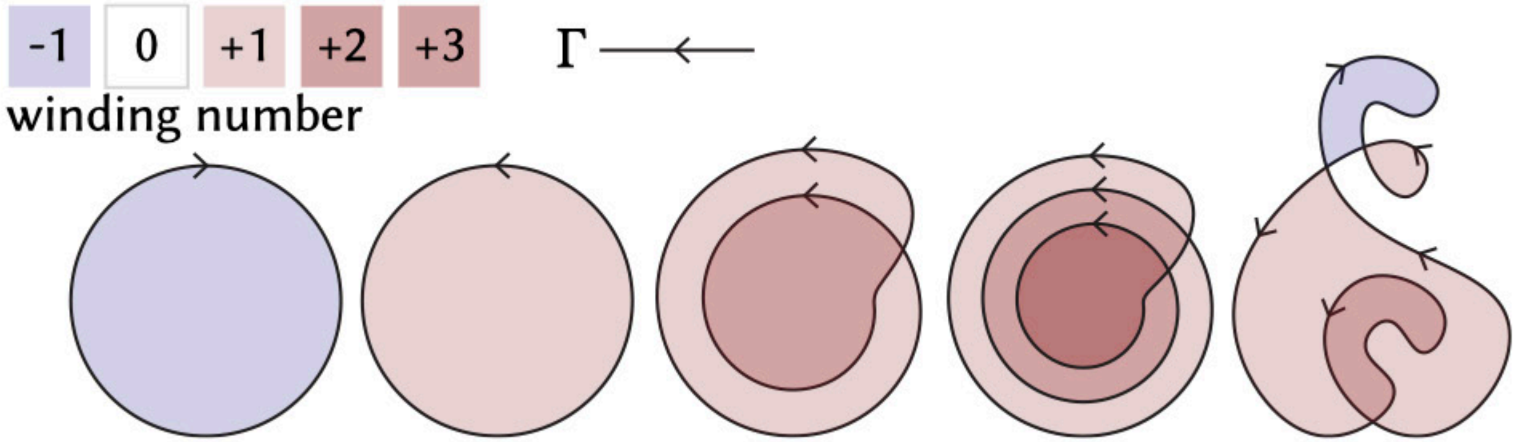

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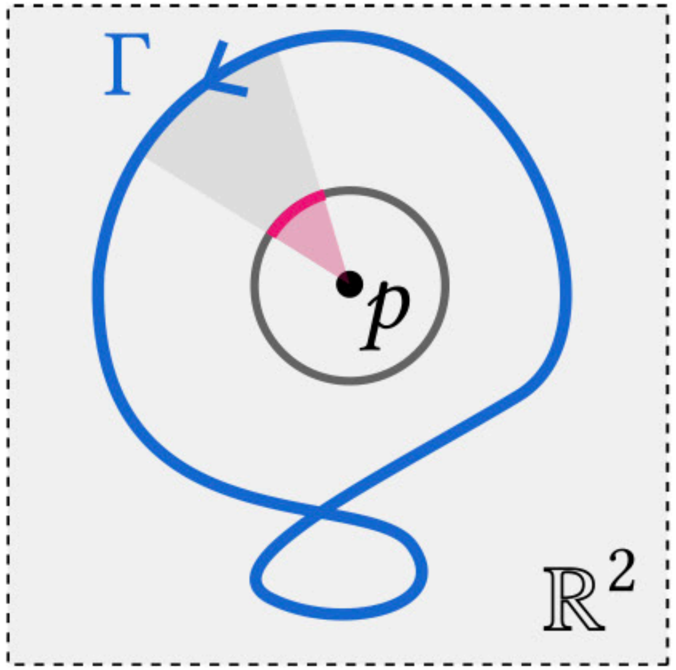

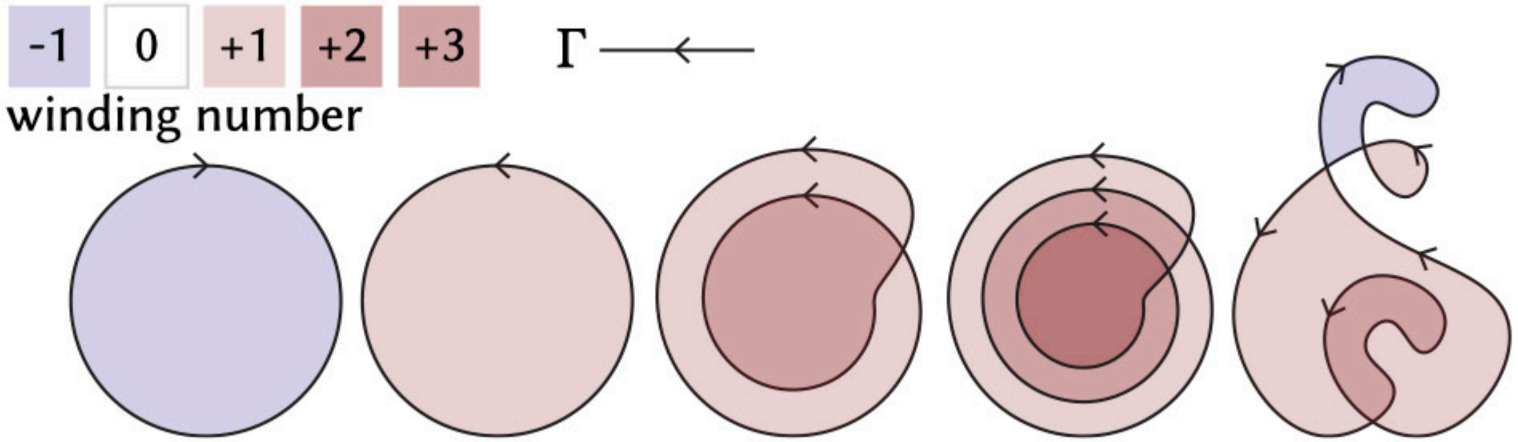

Complex analysis, differential geometry, topology, electromagnetism...

Winding numbers have succeeded before!

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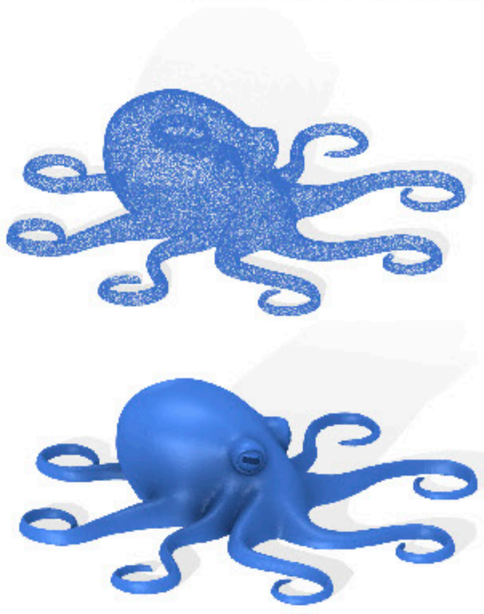
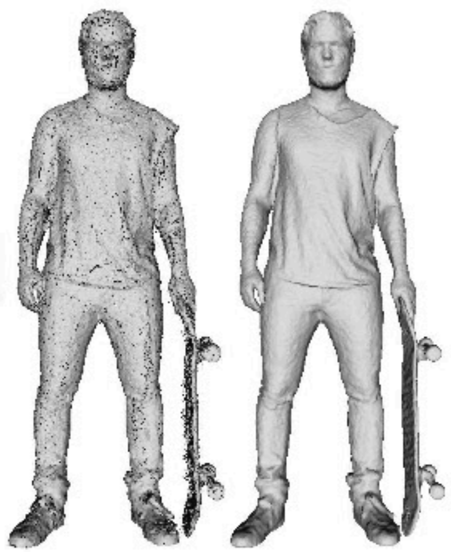
surface reconstruction

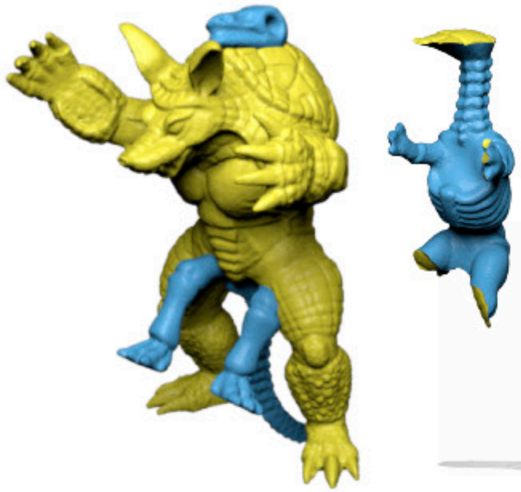


[Barill et al. 2018]

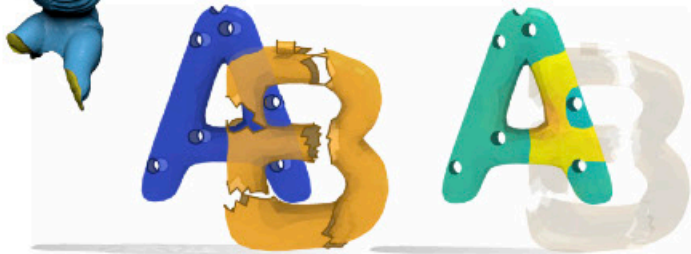


[Collet et al. 2015]

mesh booleans



[Zhou et al. 2016]



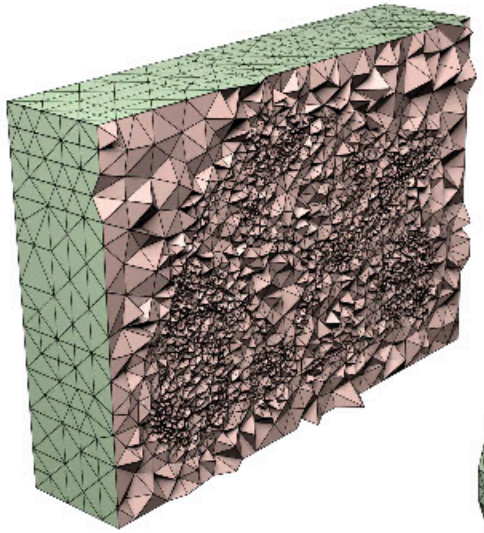
[Barill et al. 2018]

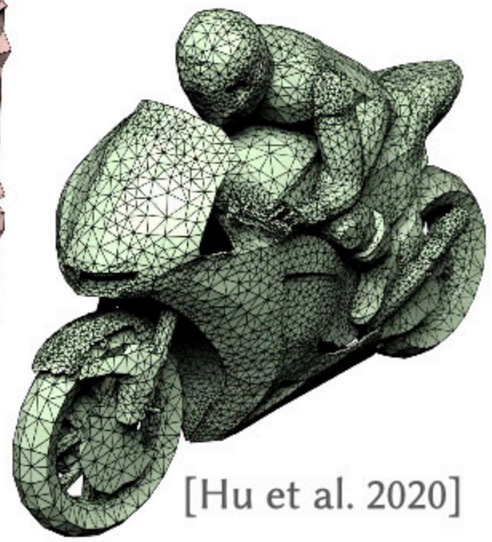
iterative normal estimation

[Xu et al. 2023]

[Hou et al. 2022]

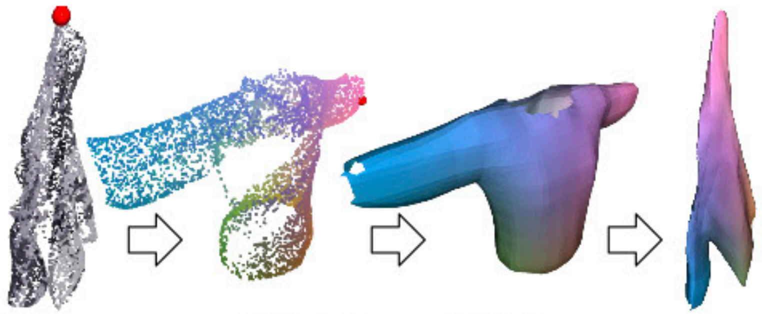
tet-meshing



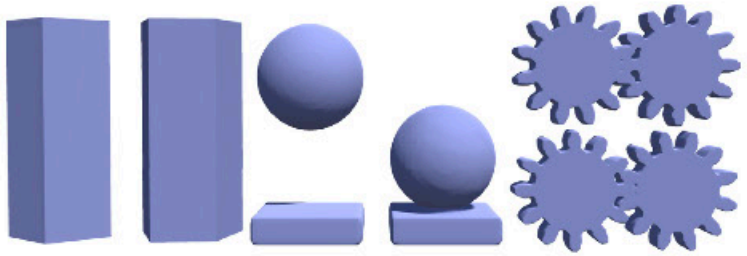


[Hu et al. 2020]

geometric preprocessing



[Chi, Song 2021]



[Dvořák et al. 2021]

Our motivation

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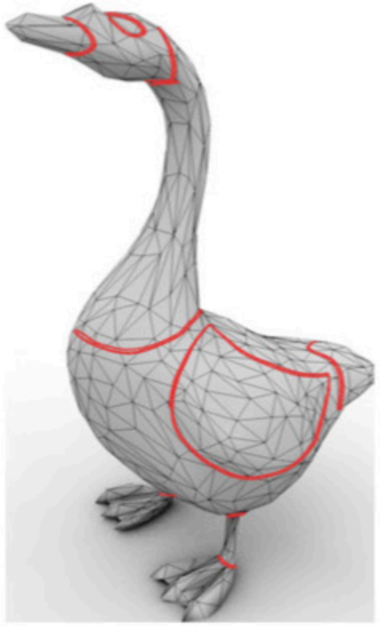

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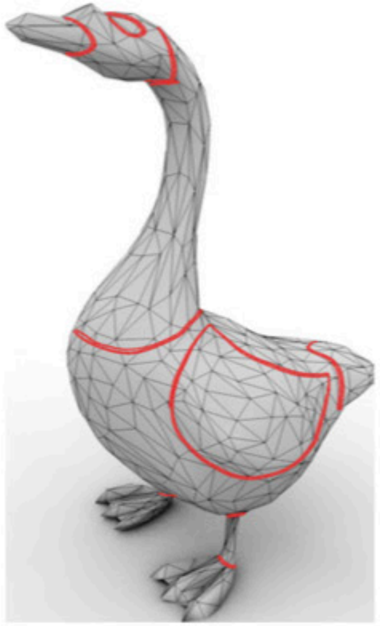

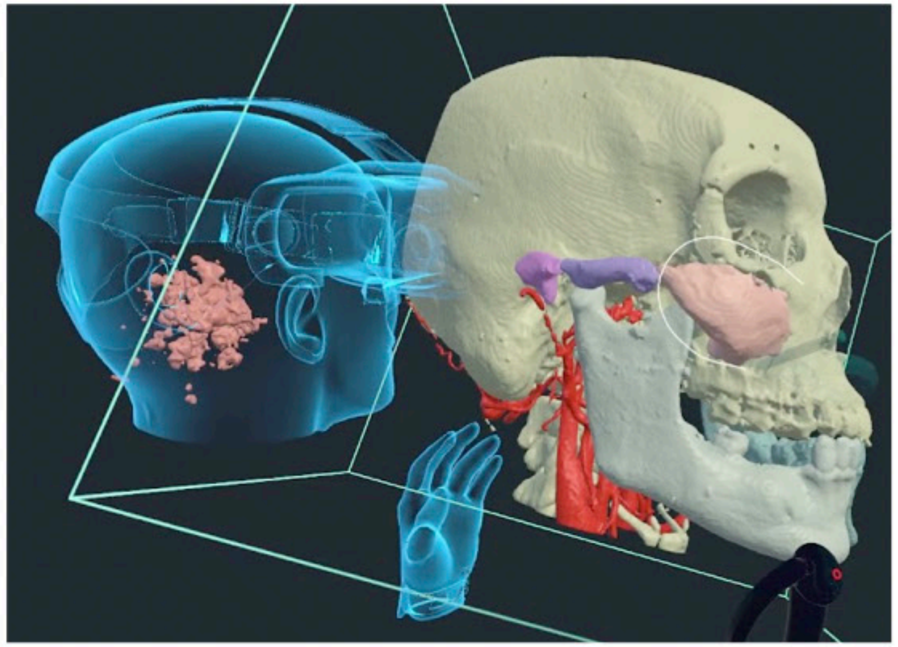

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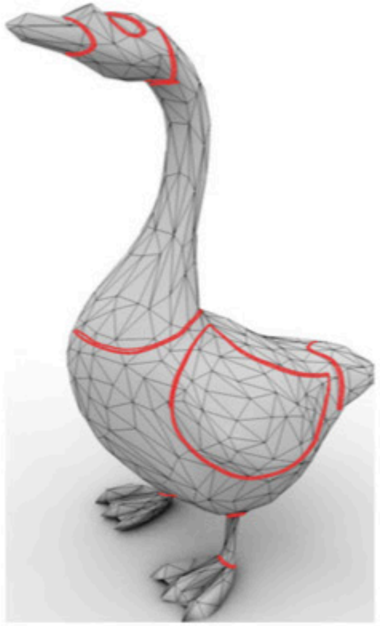

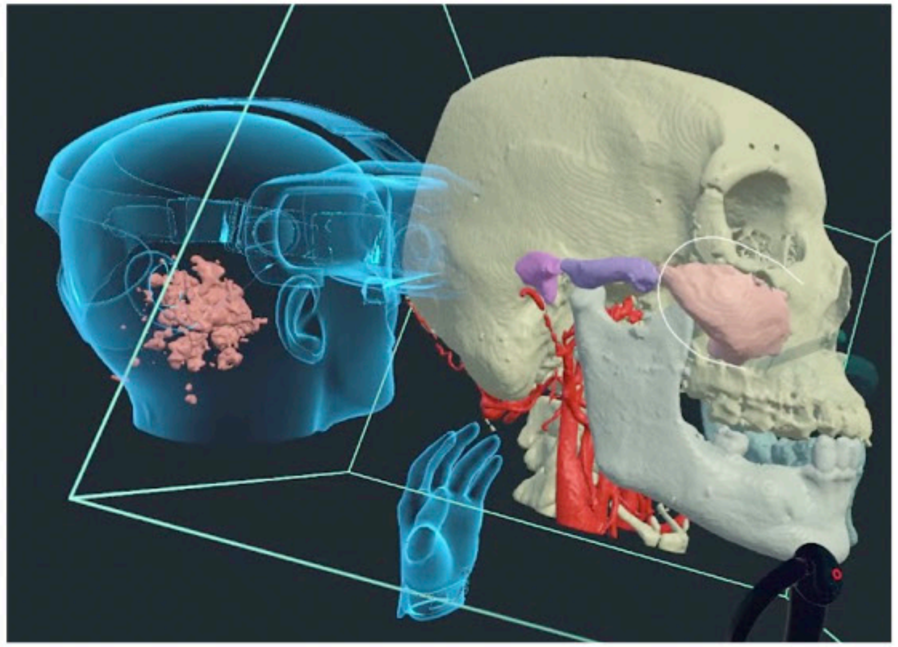

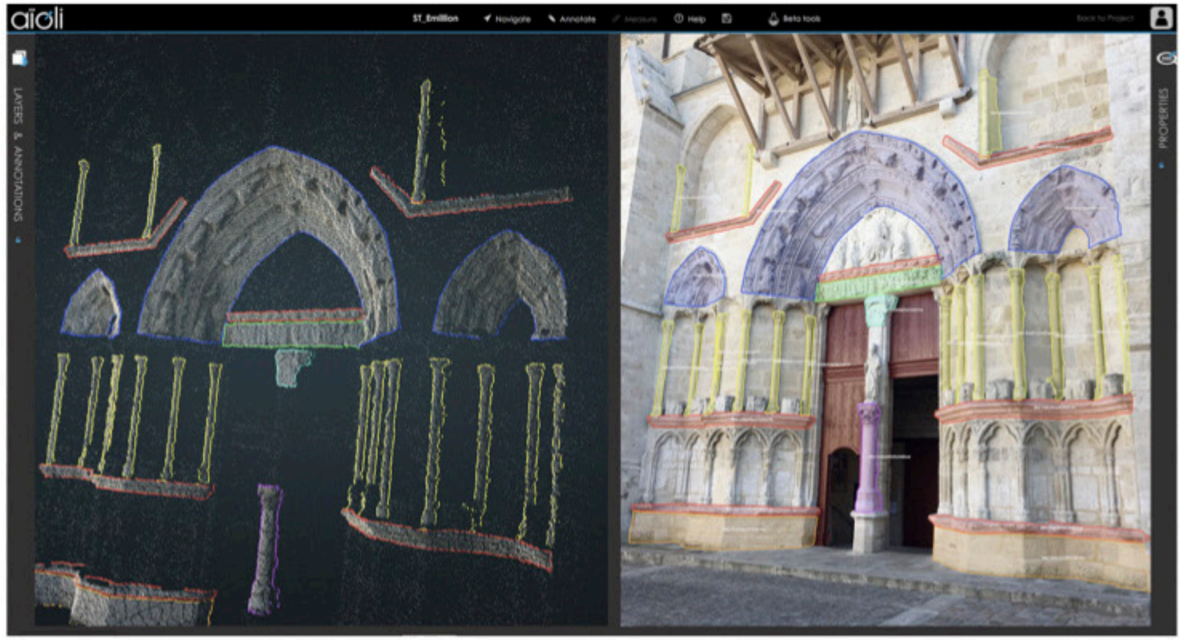

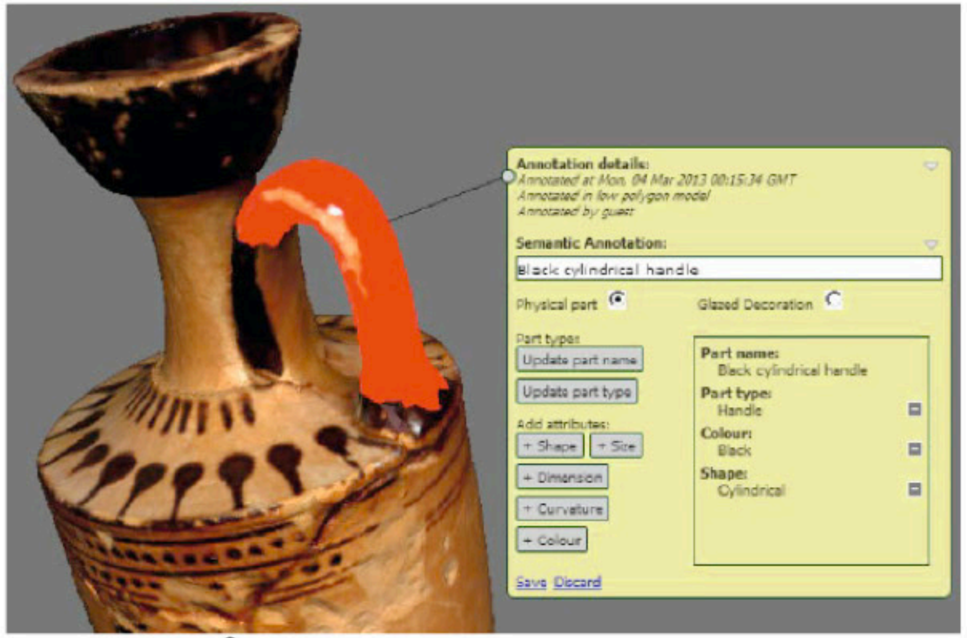

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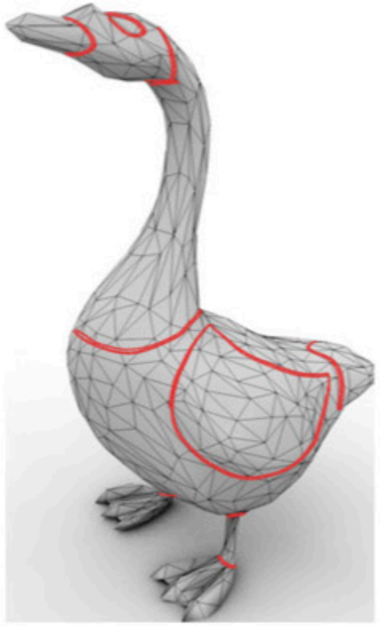

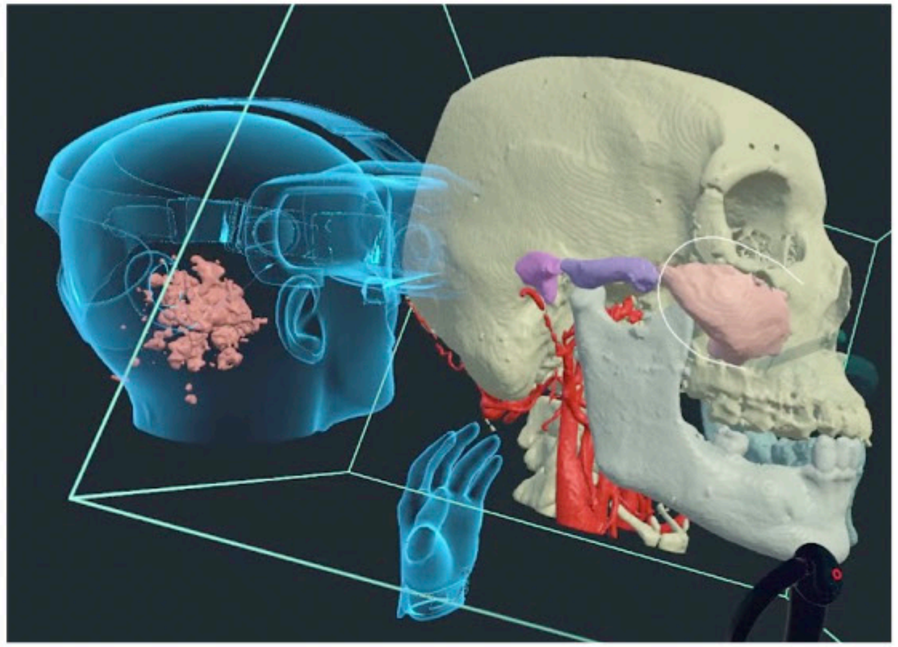

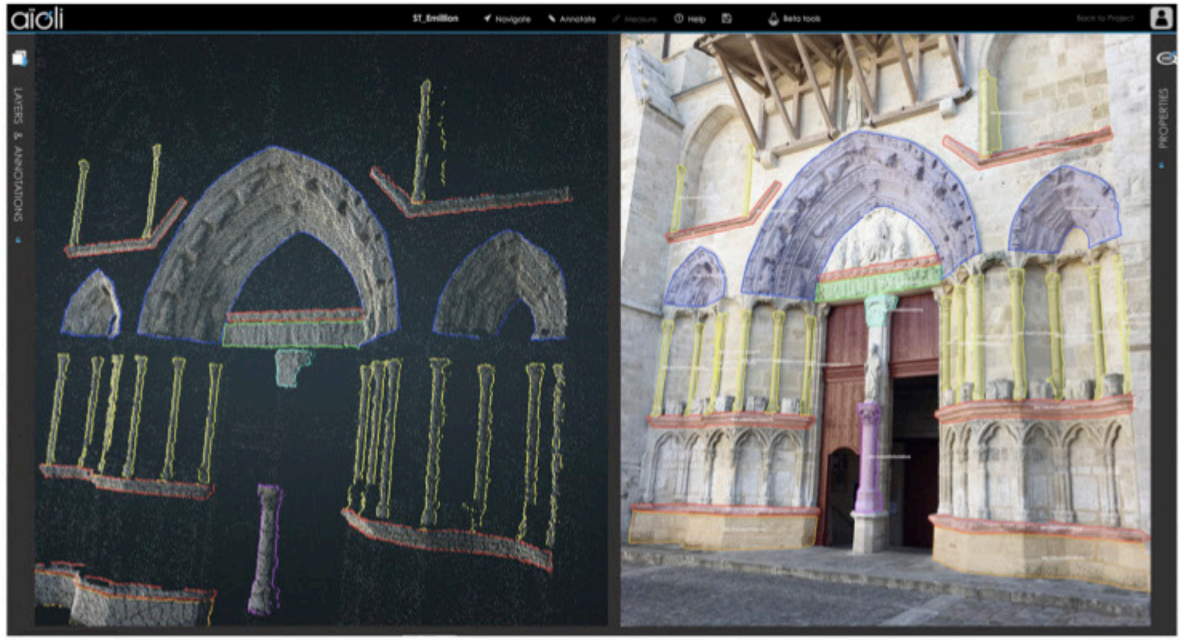

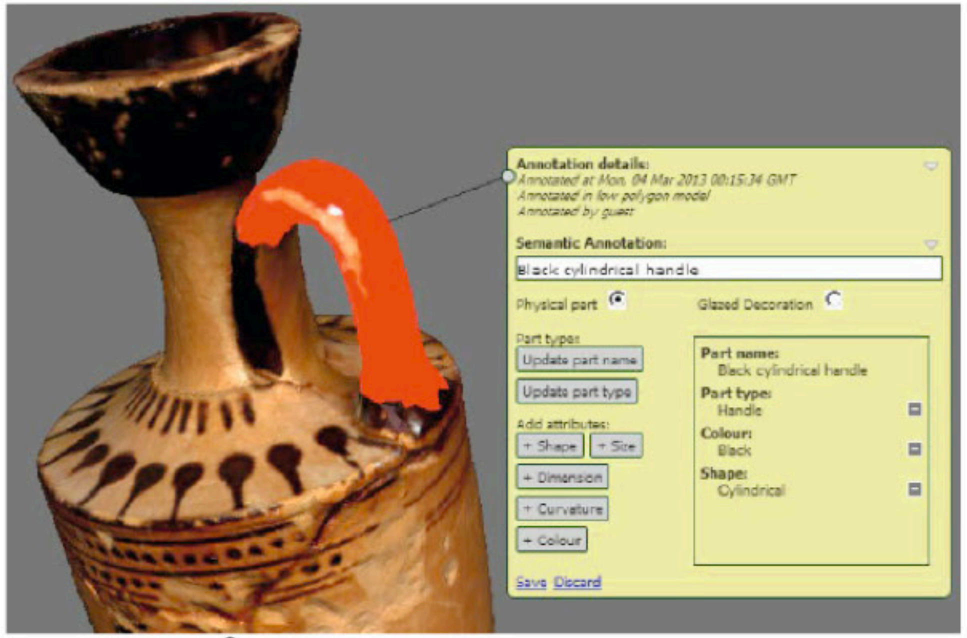

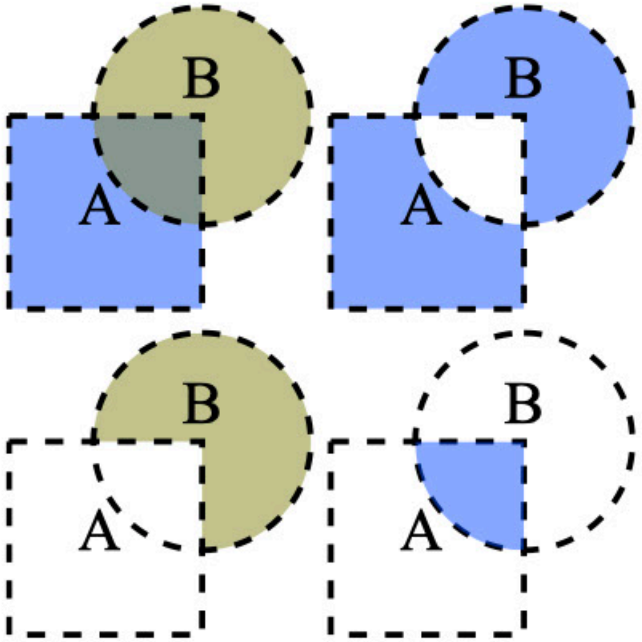

Past definitions: Euclidean only!

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(Generalized) winding number = solid angle

[Euler 1781; Lagrange 1798; Gauss 1838, Maxwell 1881...]

On Solid Angles.

417.] We have already proved that at any point P the potential due to a magnetic shell is equal to the solid angle subtended by the edge of the shell multiplied by the strength

[Maxwell 1881]

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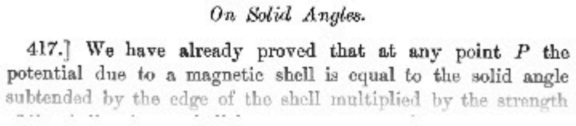
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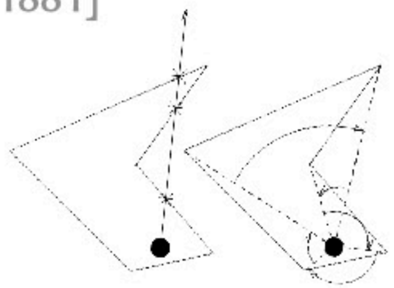
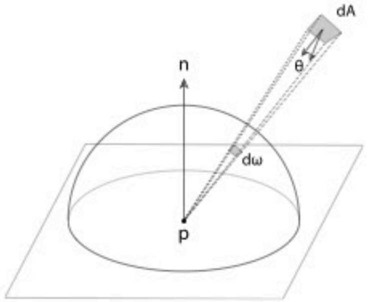
Winding number & solid angle in graphics

[Shimrat 1962; Haines 1994; Goral et al. 1984; Veach & Guibas 1995…]



[Haines 1994]



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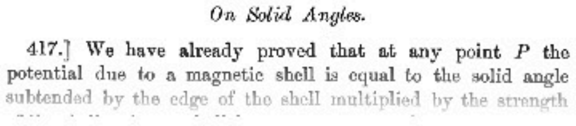
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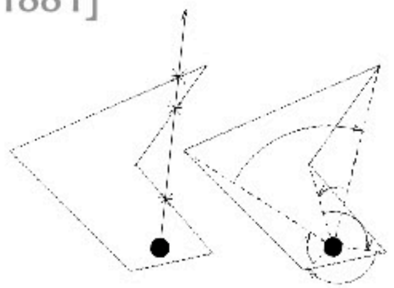

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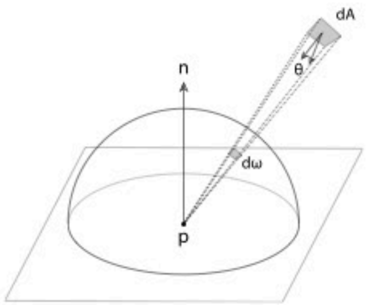
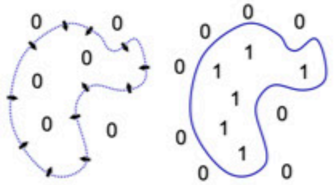
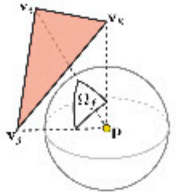


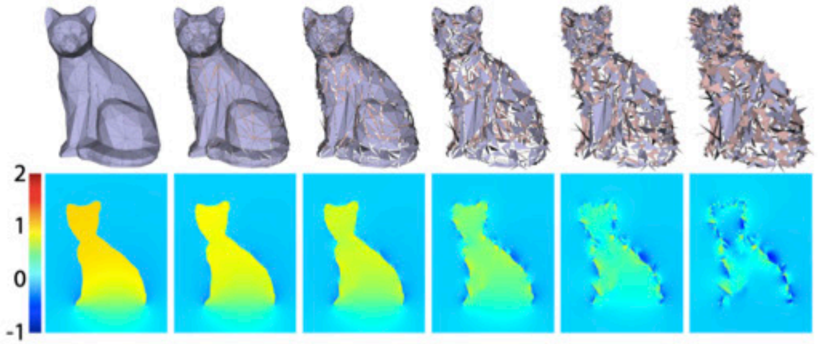
[Pharr et al. 2018]

Poisson Surface Reconstruction = Generalized Winding Number

[Kazhdan et al. 2006] [Jacobson et al. 2013]







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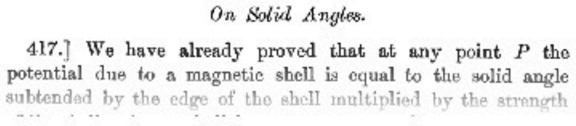
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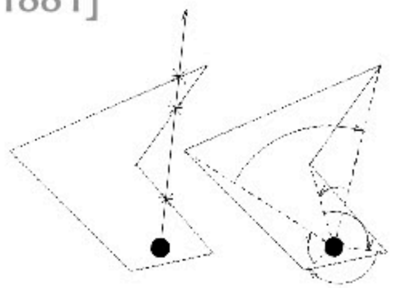

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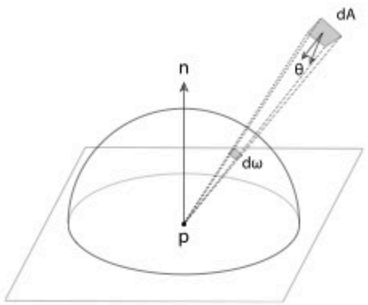

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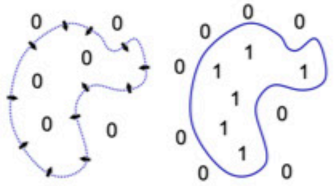

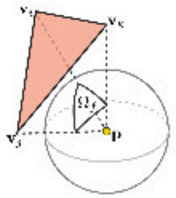

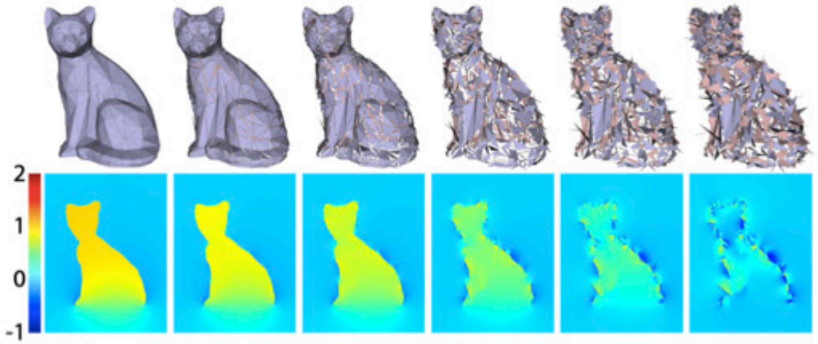

[Jacobson et al. 2013]

~~Winding~~ Turning number on surfaces

[Reinhart 1960, 1963; Chillingworth 1972; Humphries & Johnson 1989; McIntyre & Cairns 1993; Chernov & Rudyak 2009]

Classic methods fail

Poisson Surface Reconstruction (PSR) & *Generalized Winding Number (GWN)*
don't always identify well-defined regions.

Poisson Surface Reconstruction. Kazhdan, Bolitho, Hoppe (2006)
Robust Inside-Outside Segmentation using Generalized Winding Numbers. Jacobson, Kavan, Sorkine-Hornung (2013)

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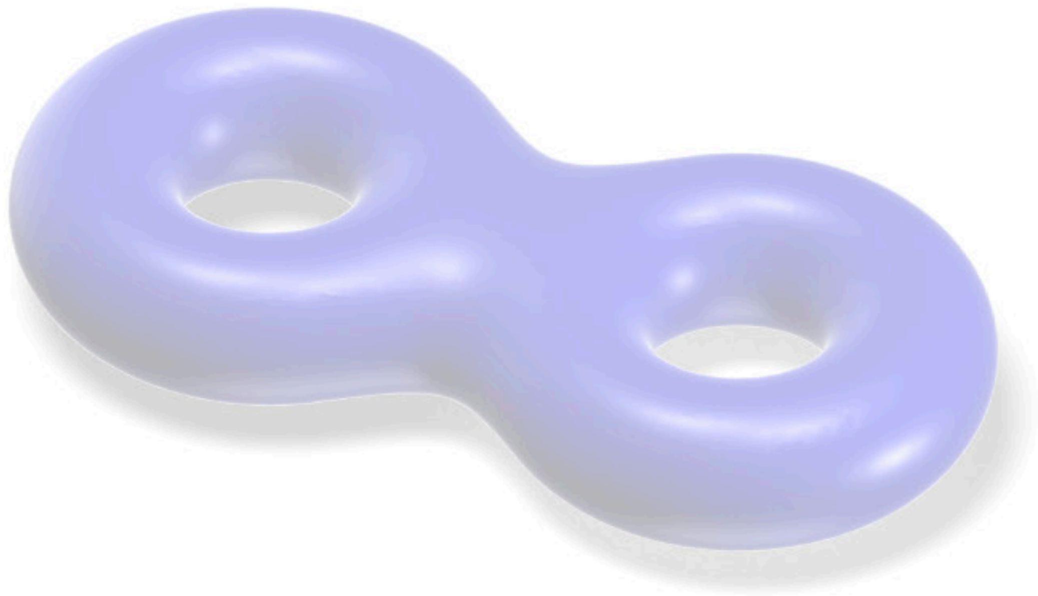

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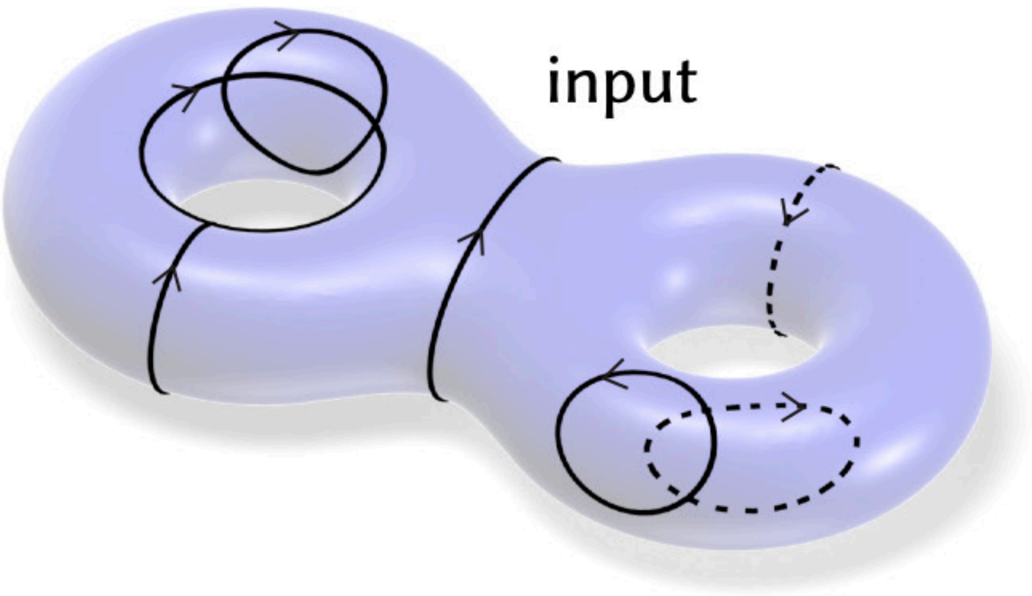

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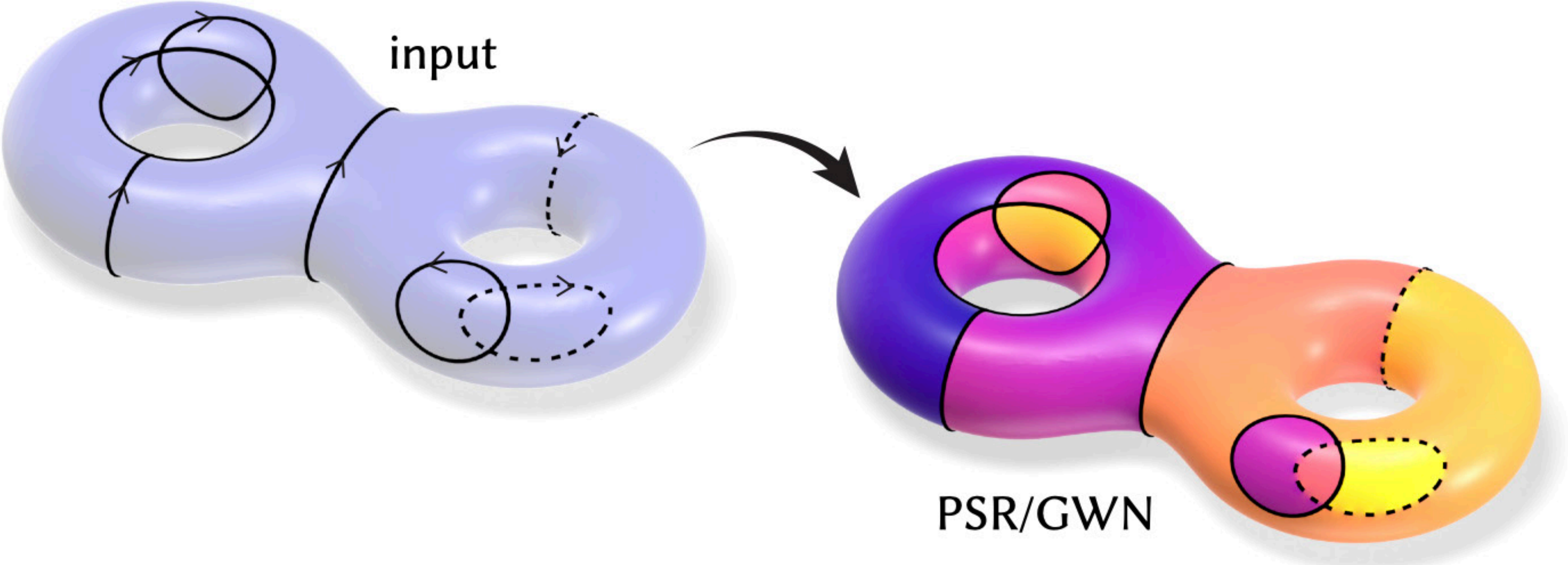

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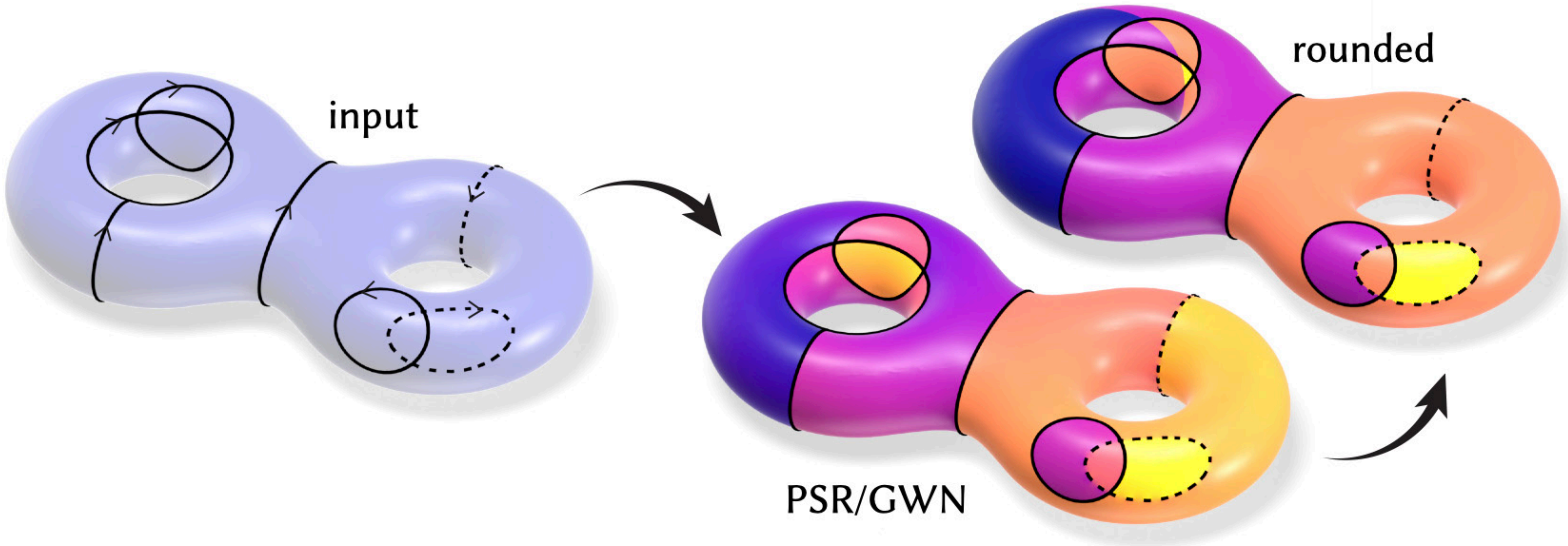

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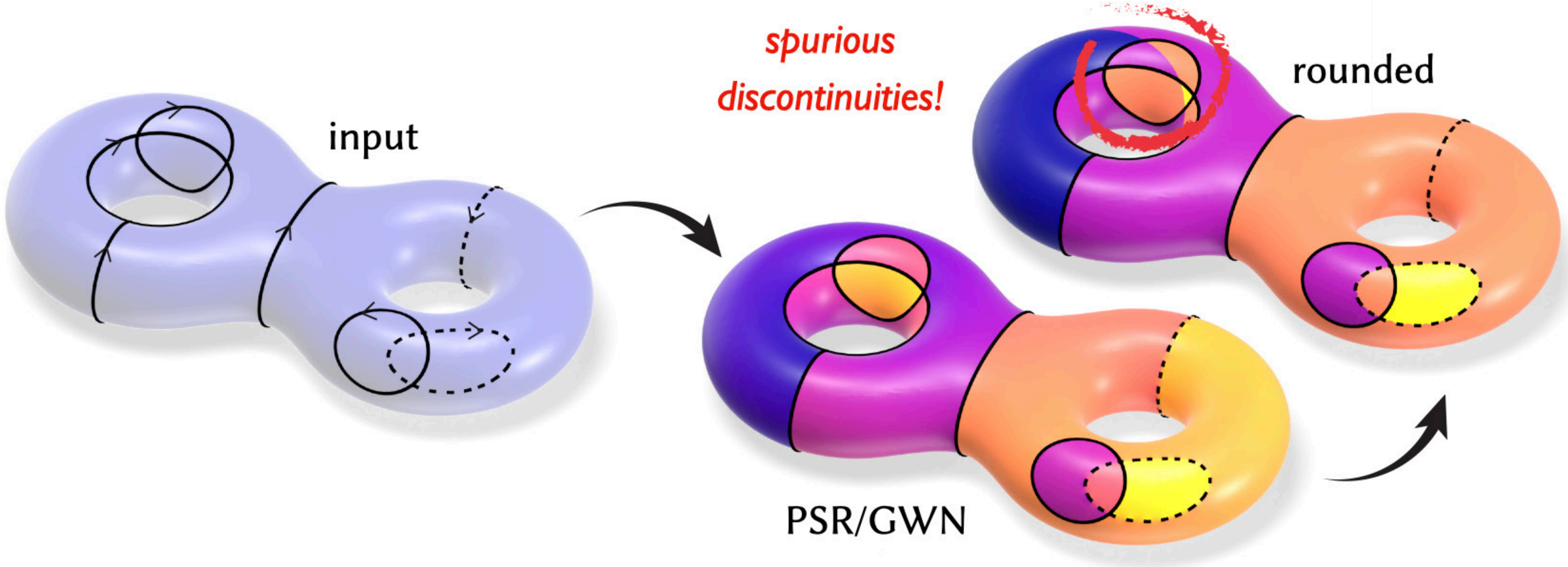

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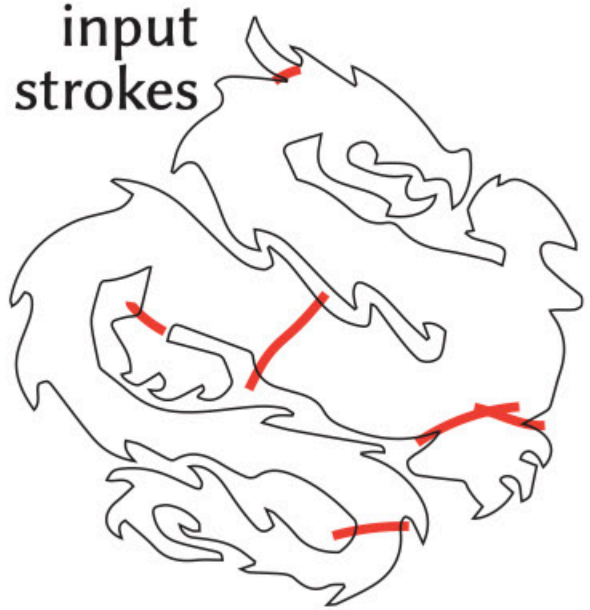

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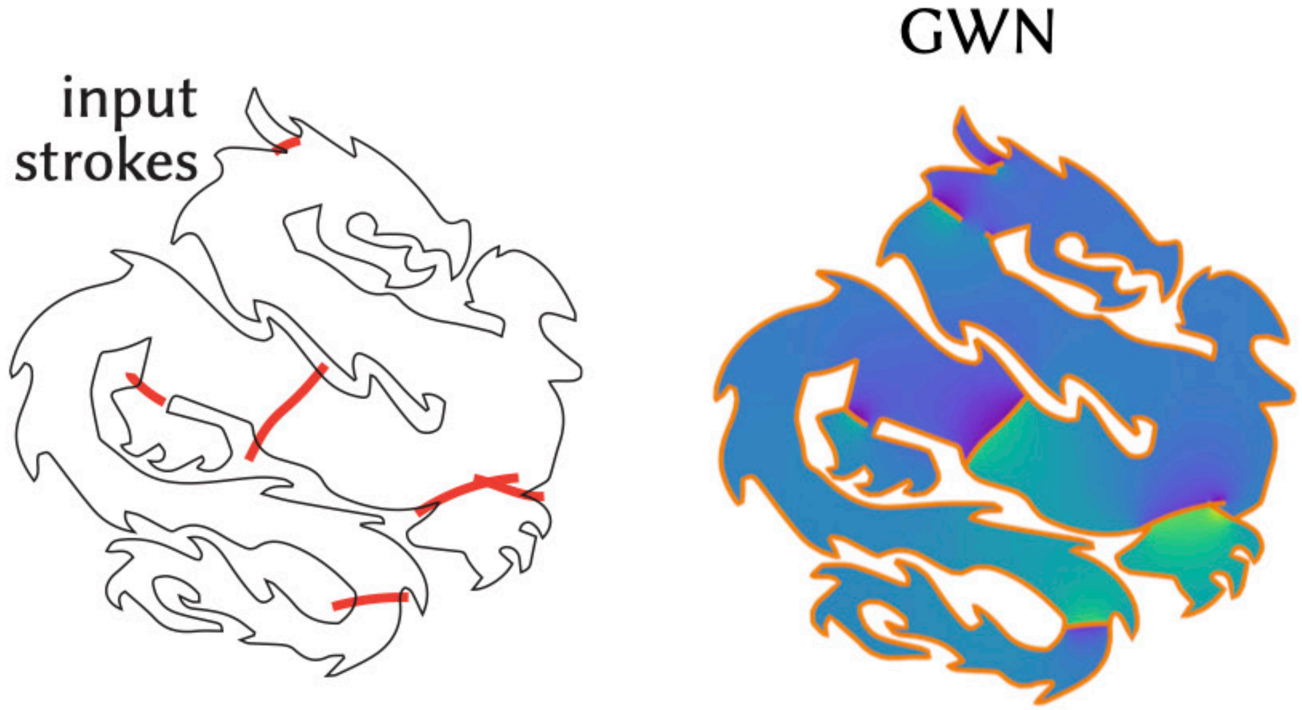

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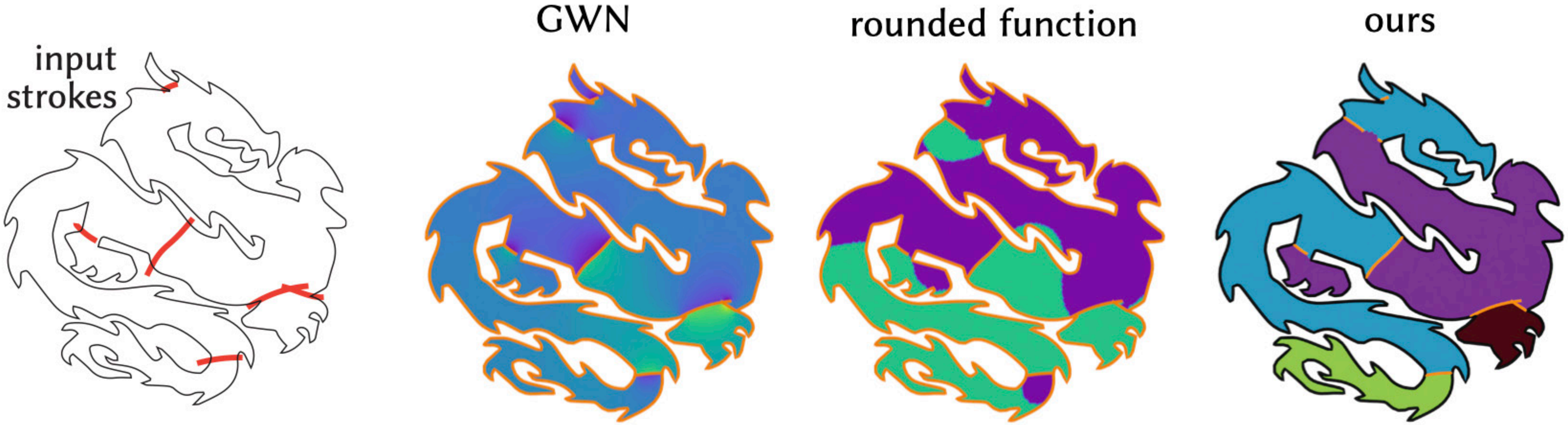

Robust Inside-Outside Segmentation using Generalized Winding Numbers.
Jacobson, Kavan, Sorkine-Hornung (2013)

Many formulas for solid angle…

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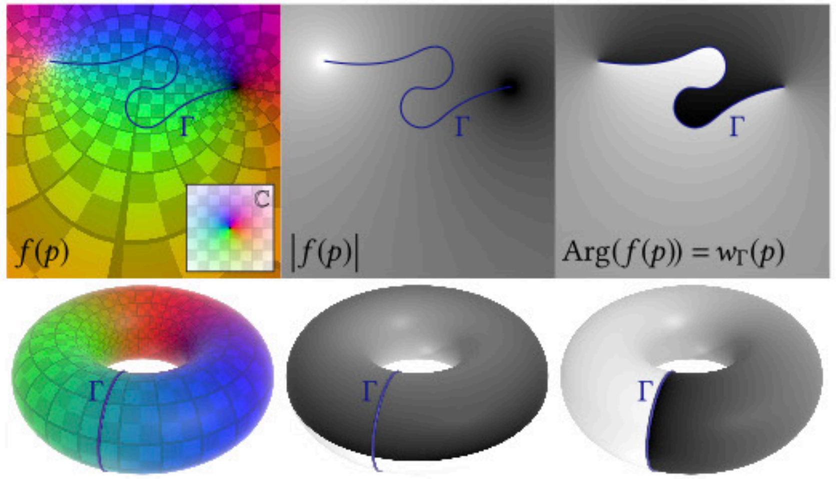

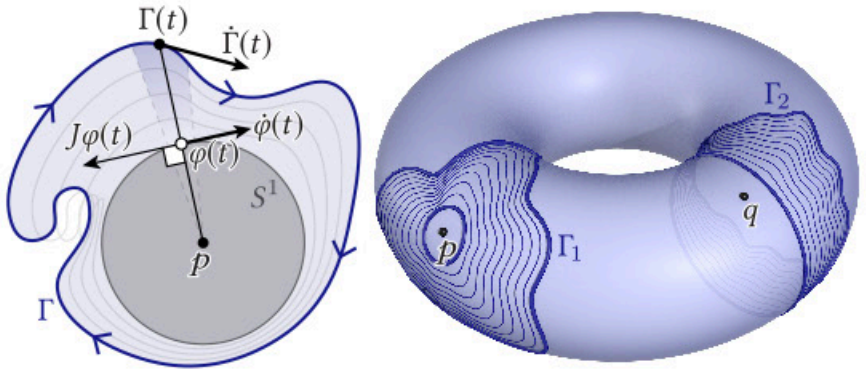

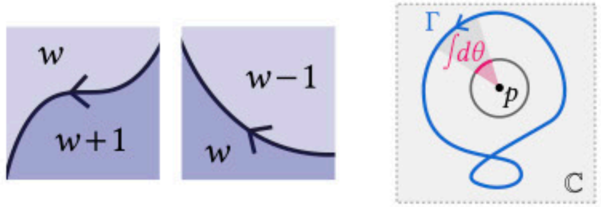

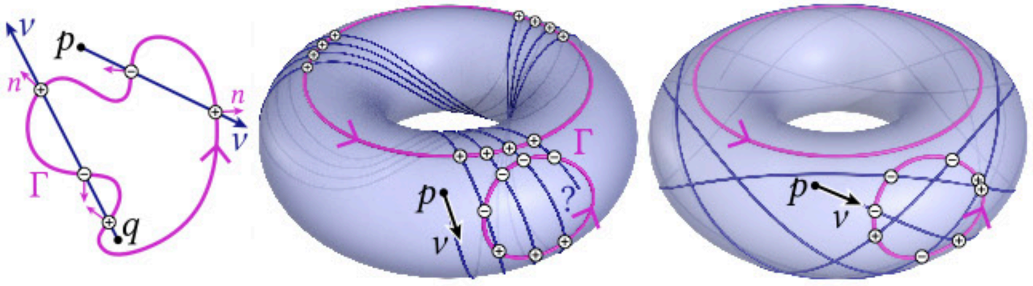

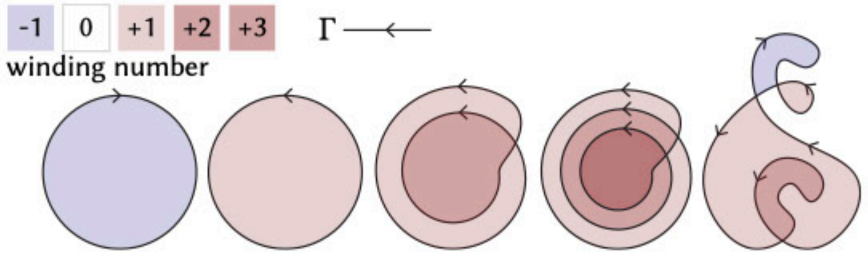

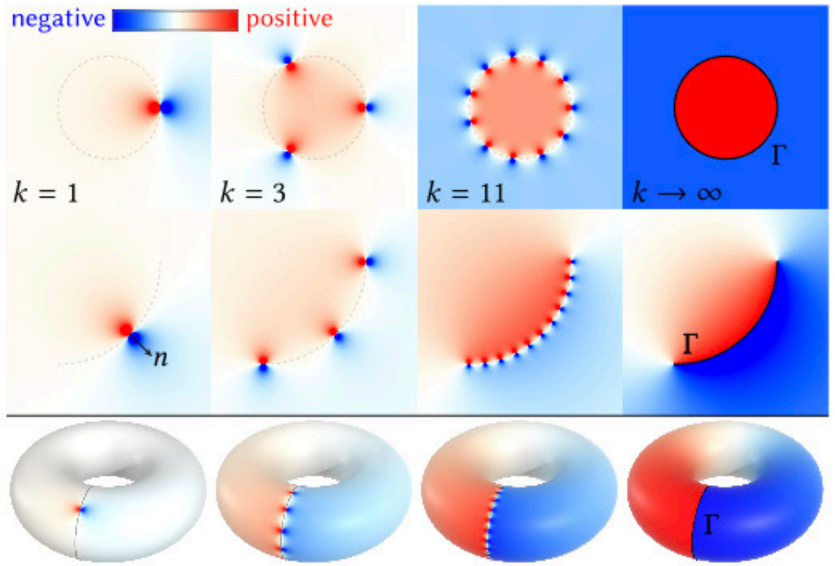

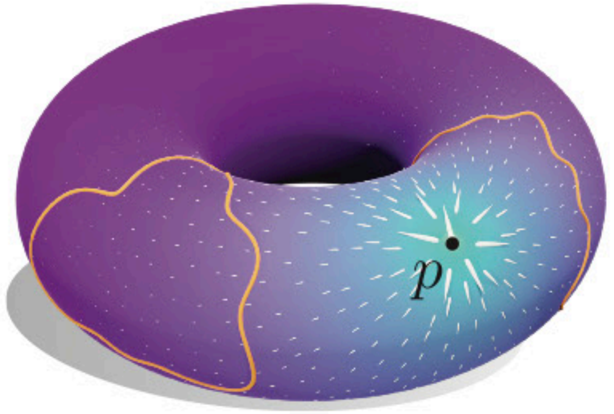

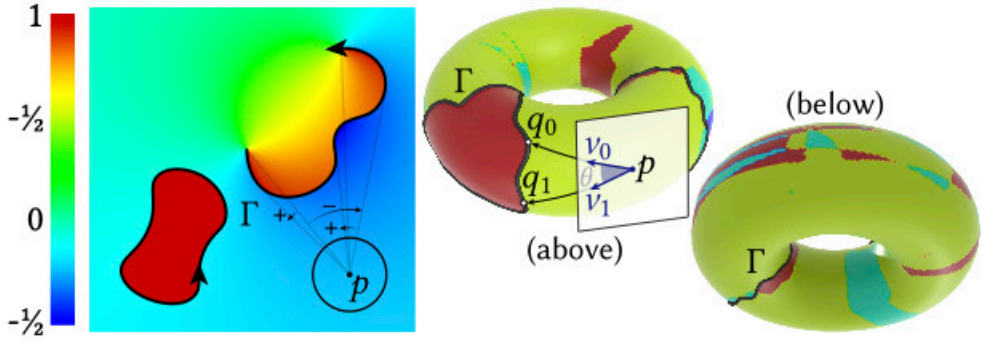

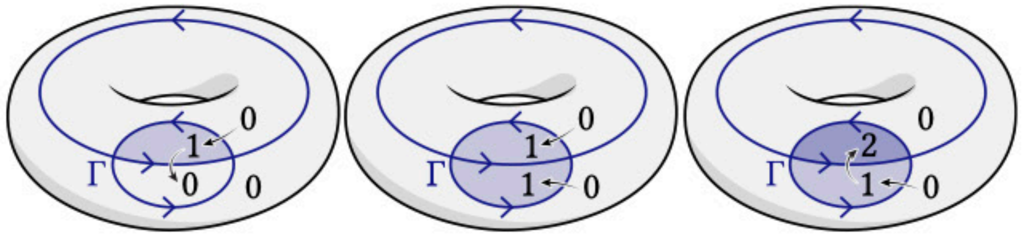

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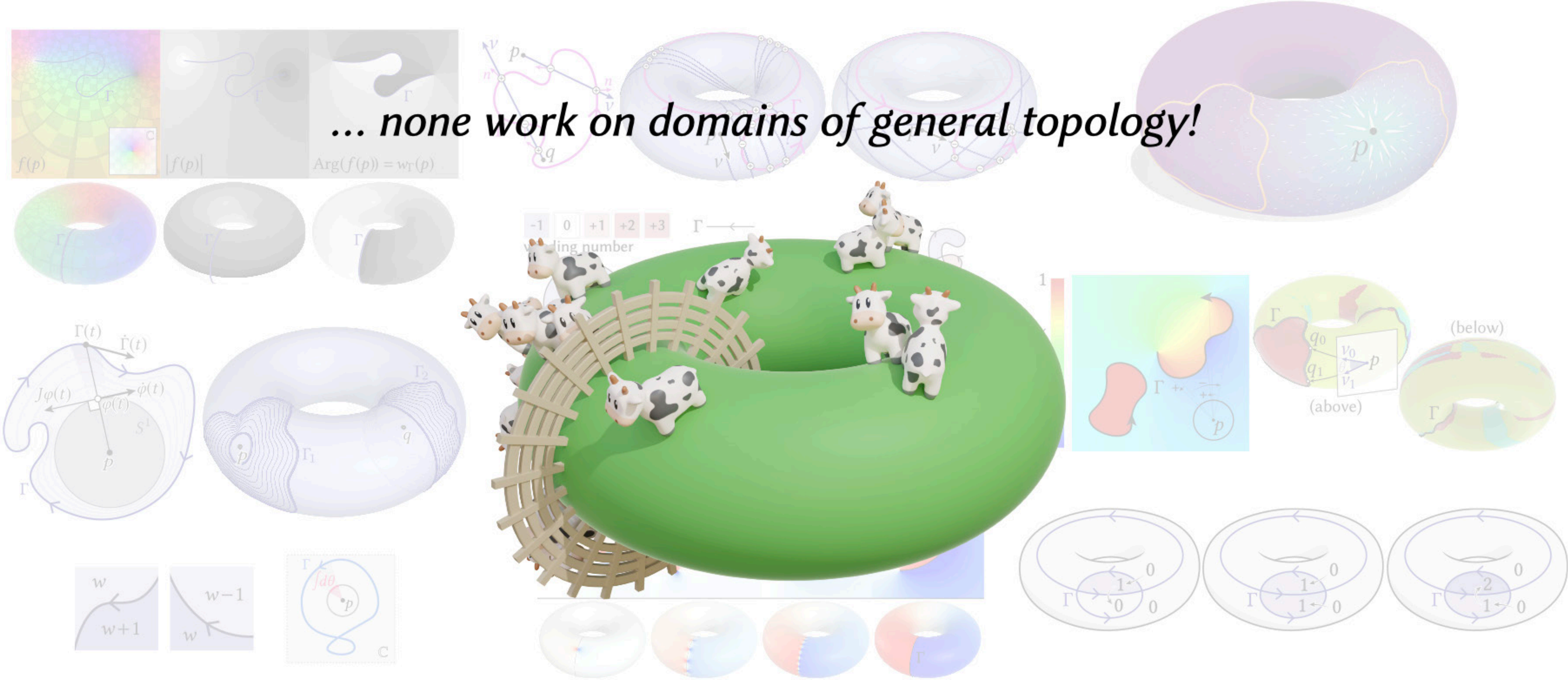

… none work on domains of general topology!

The general case is difficult!

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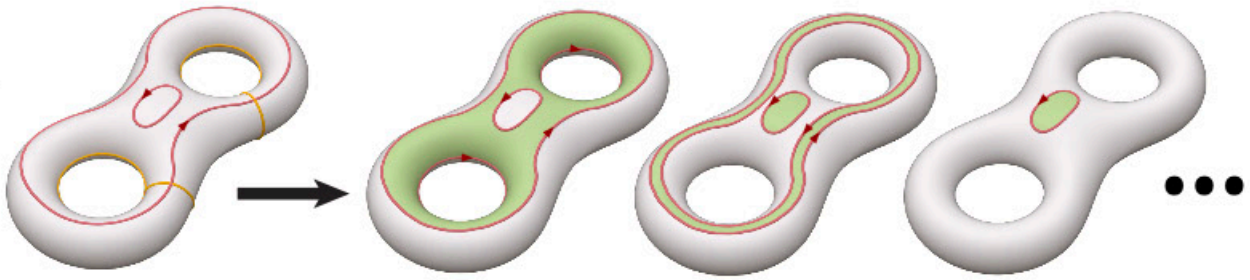

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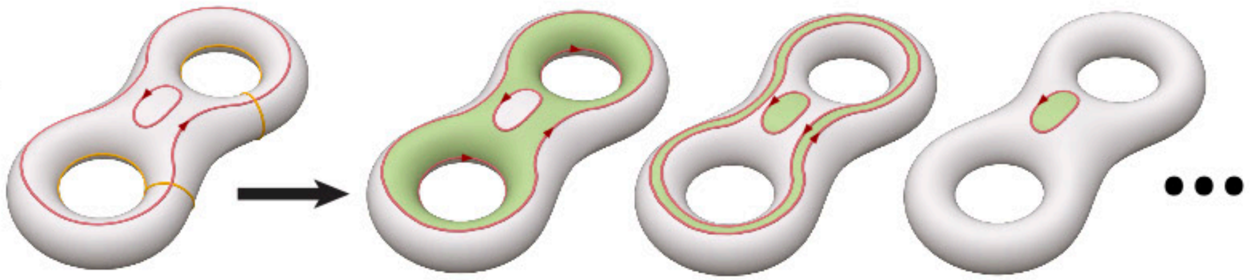

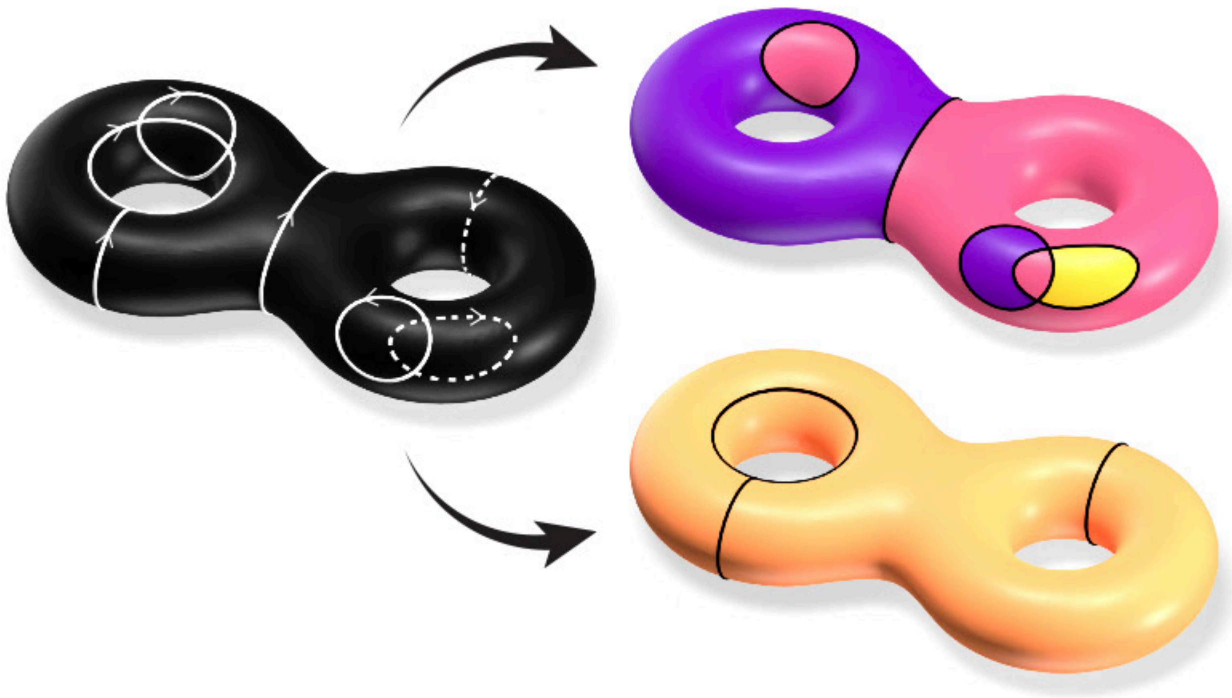

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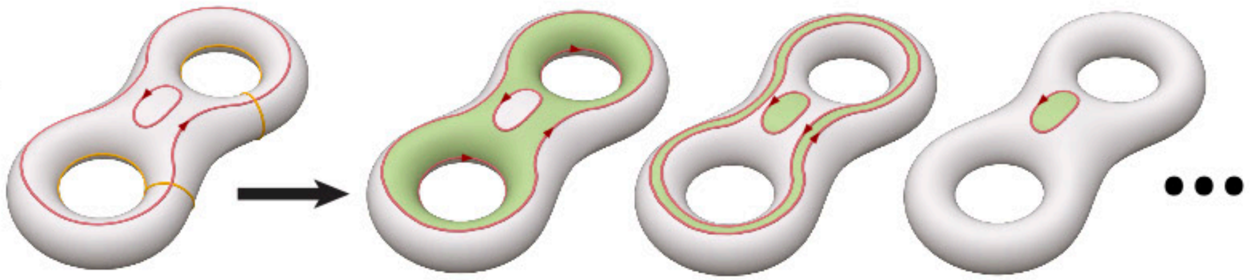

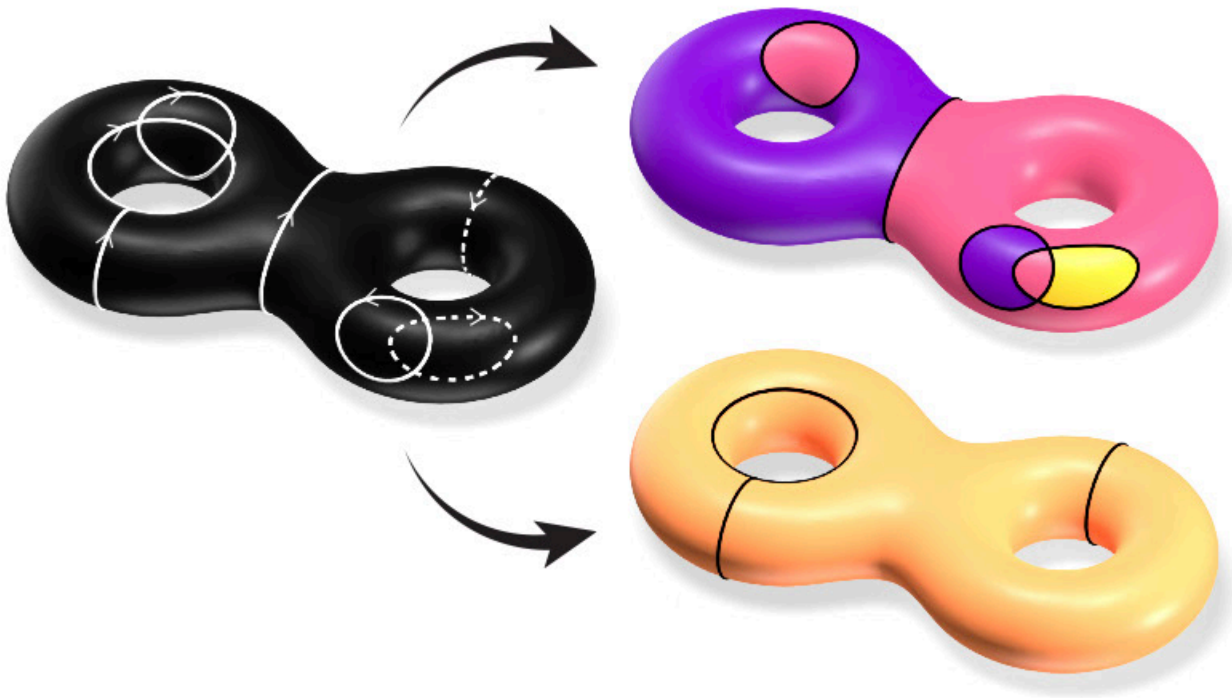

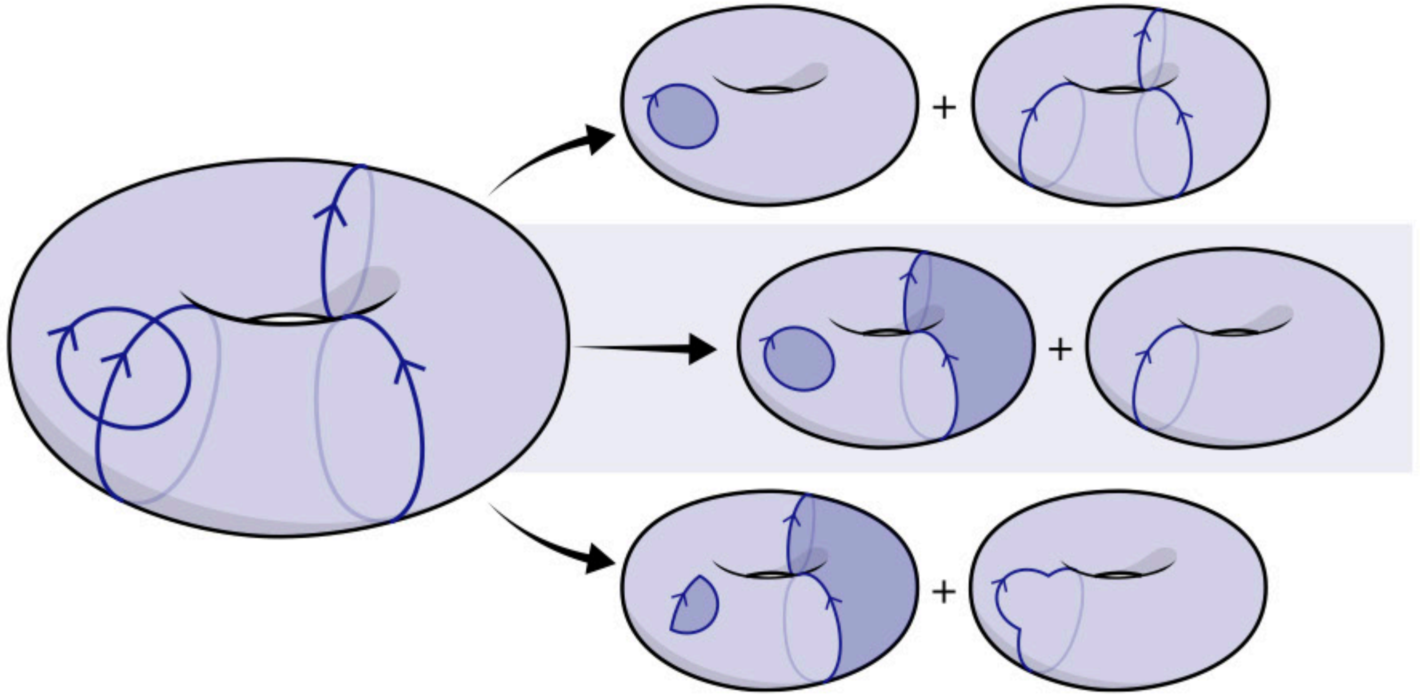

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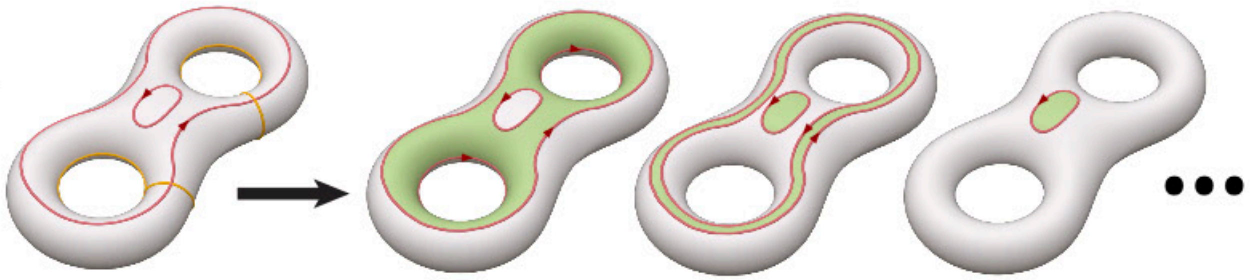

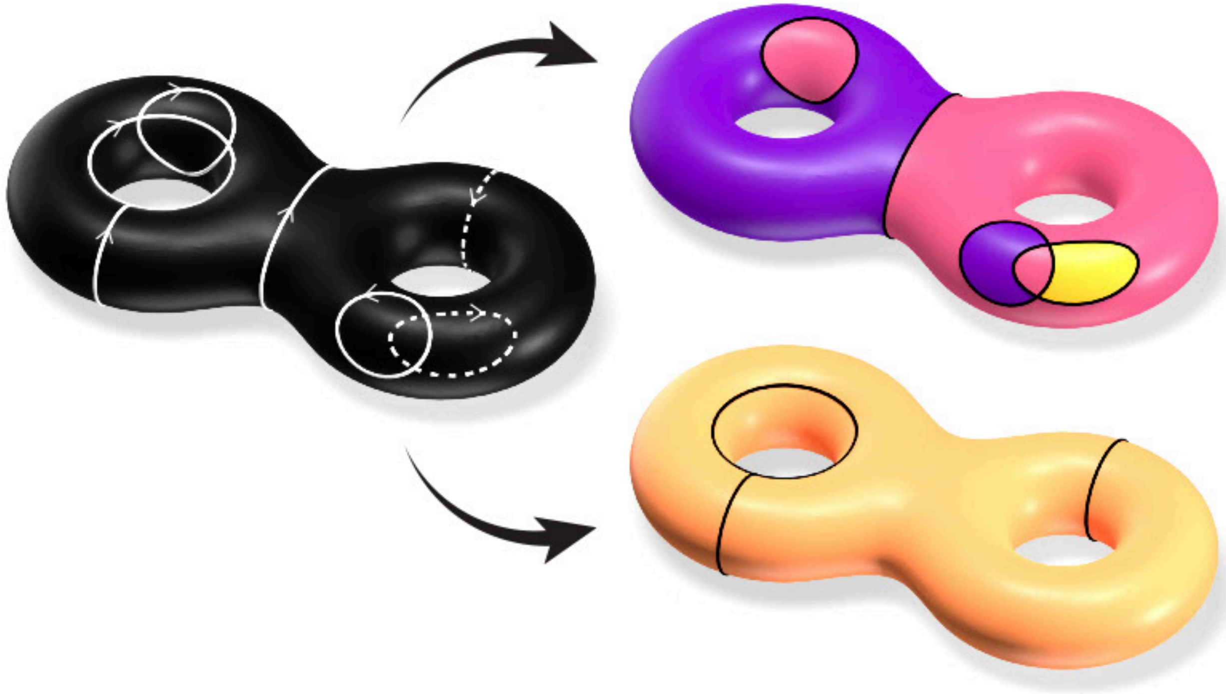

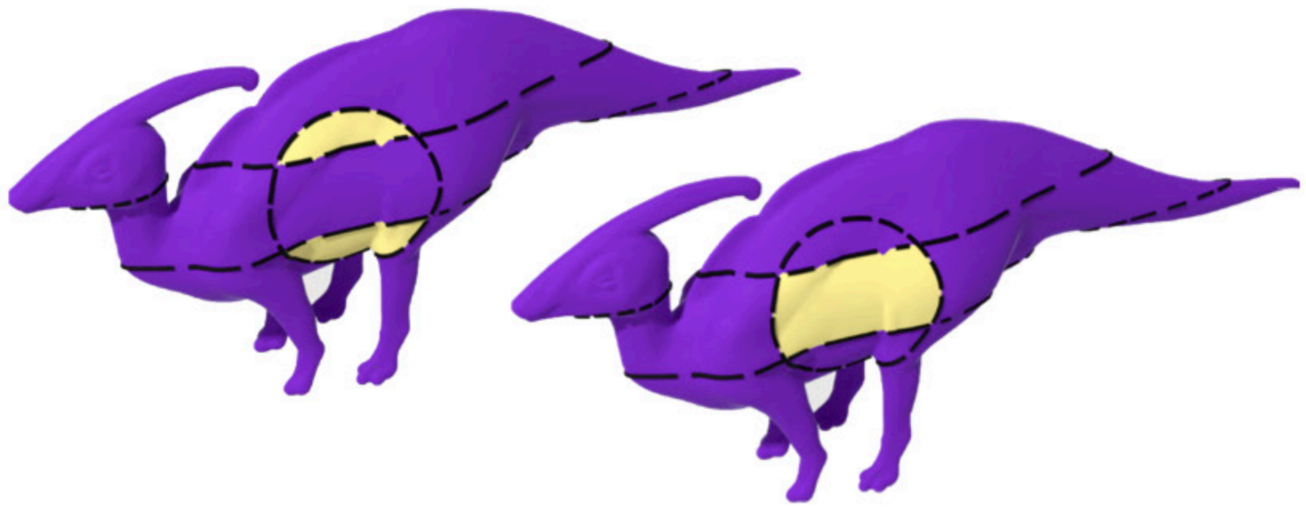

Past approaches to "homological geometry processing"

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Geometry processing with homological constraints

[Born et al. 2021; Dey et al. 2010; Wang & Chern 2021]

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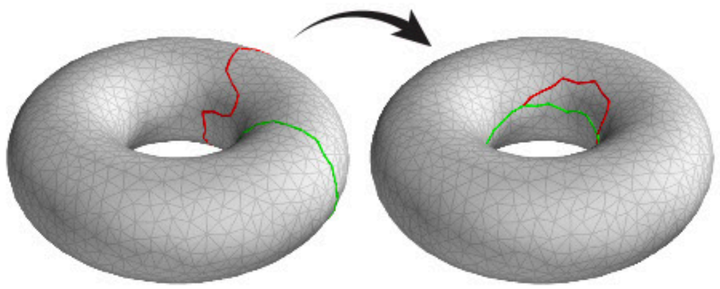

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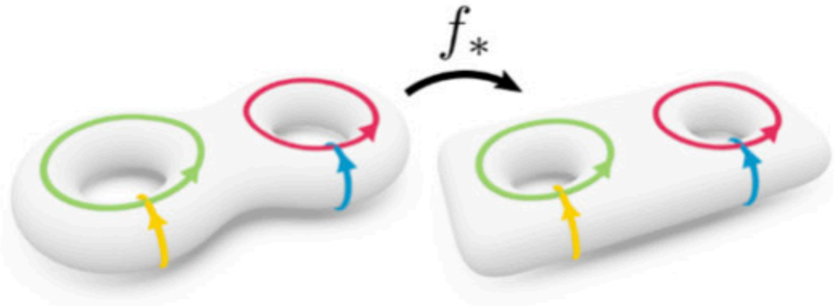

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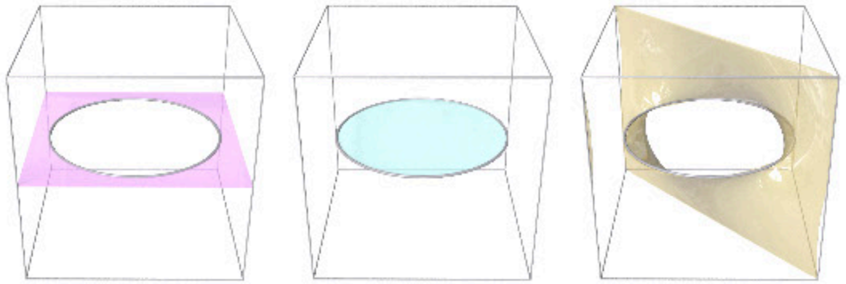

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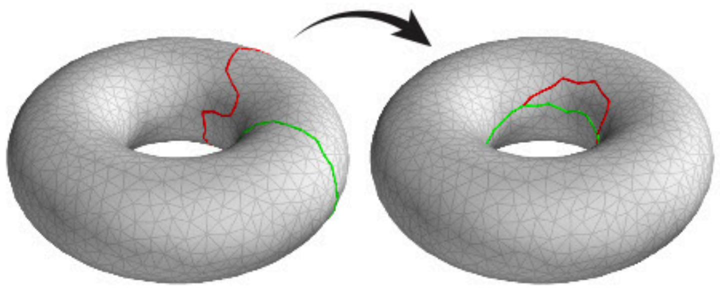

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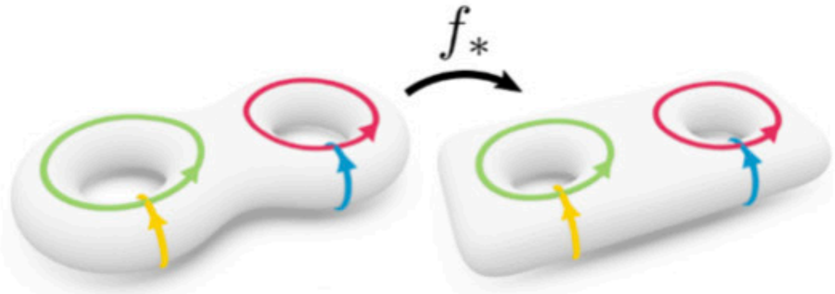

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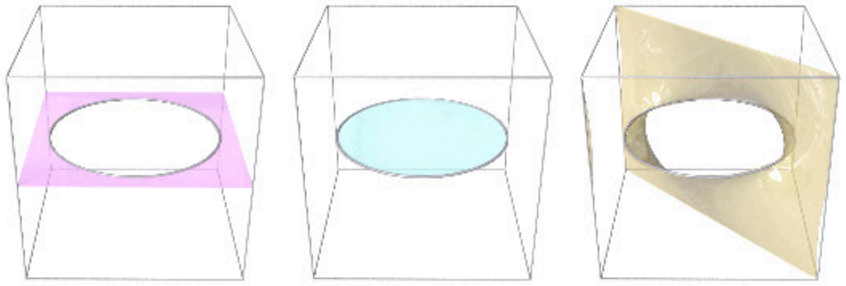
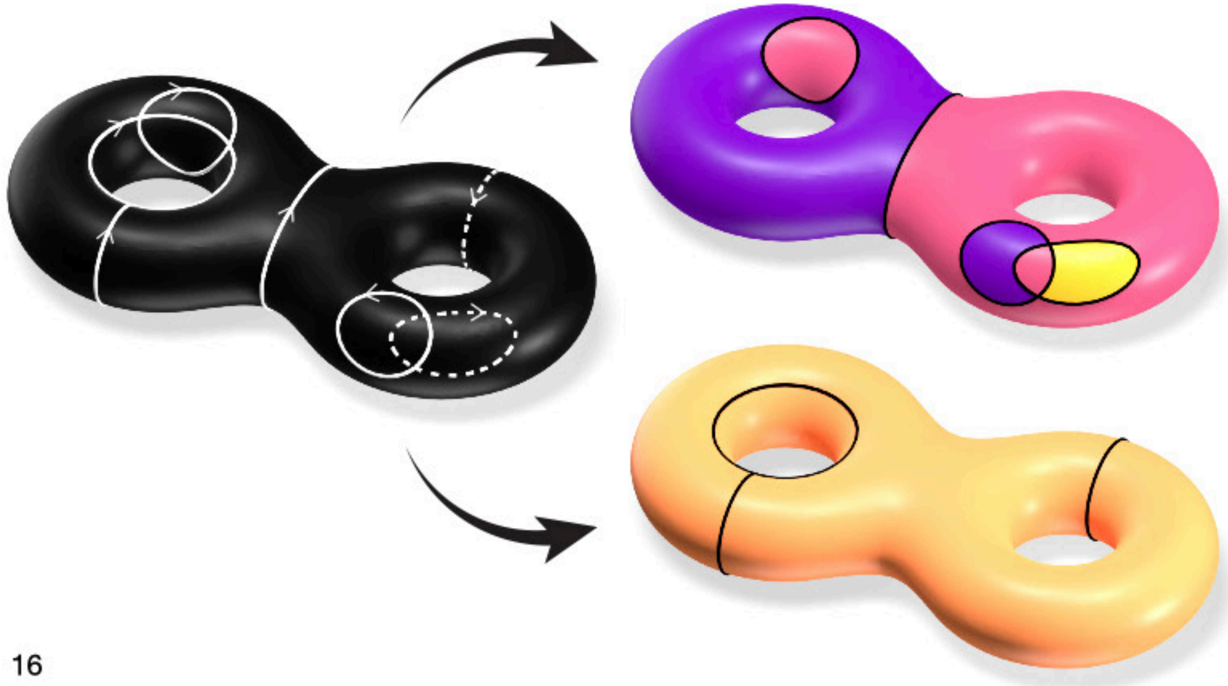


[Wang & Chern 2021]

We want to *infer* curve topology!



Algorithm input & output

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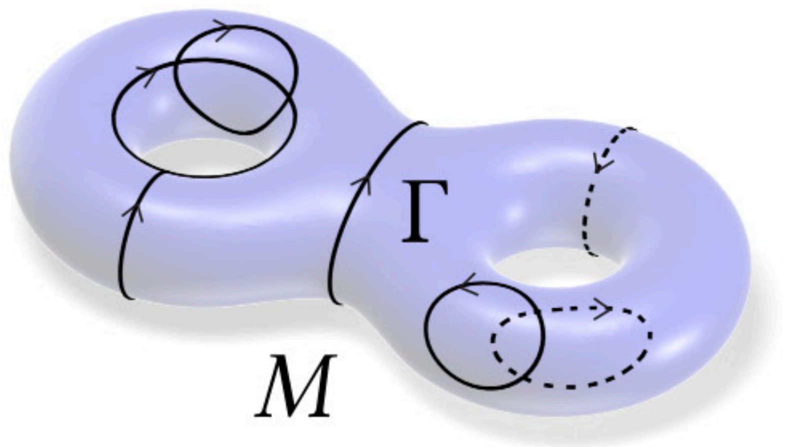
Input:

(Possibly broken) oriented curve Γ on a surface M.



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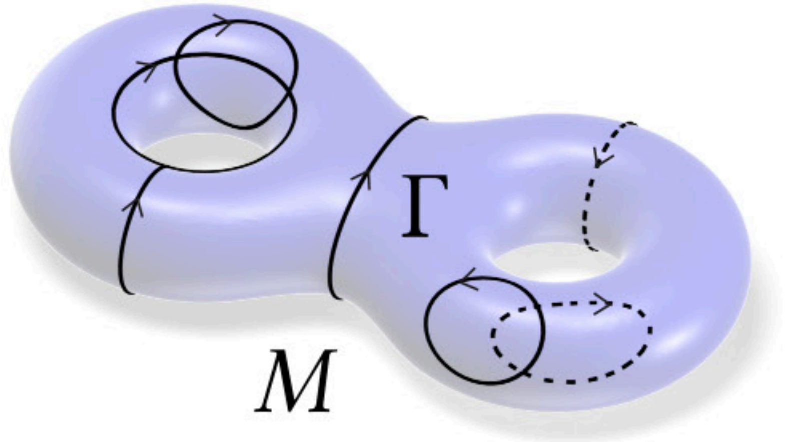

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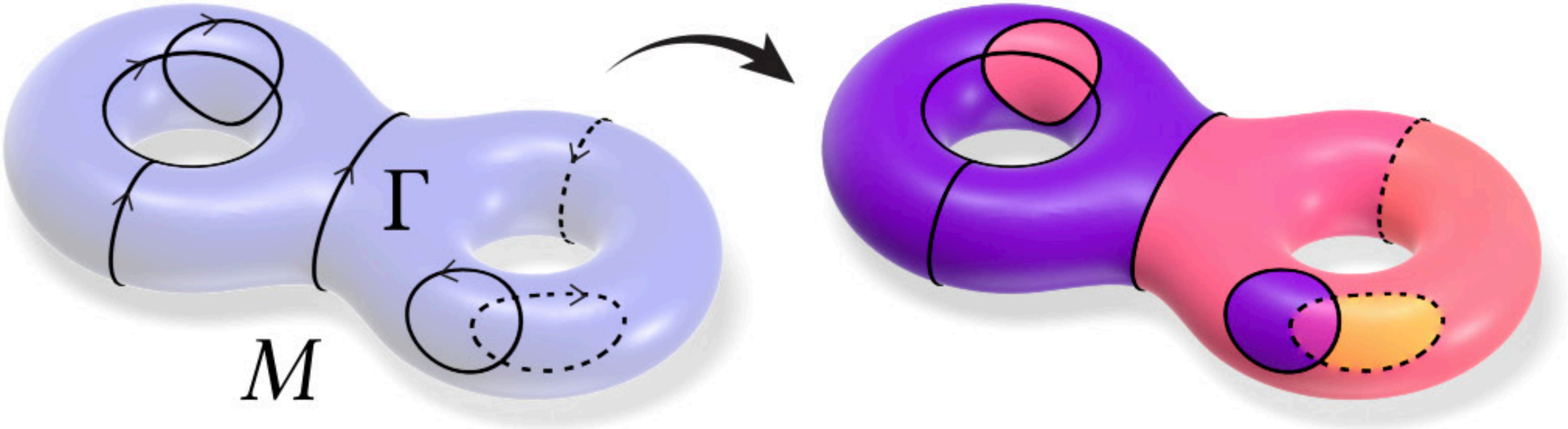

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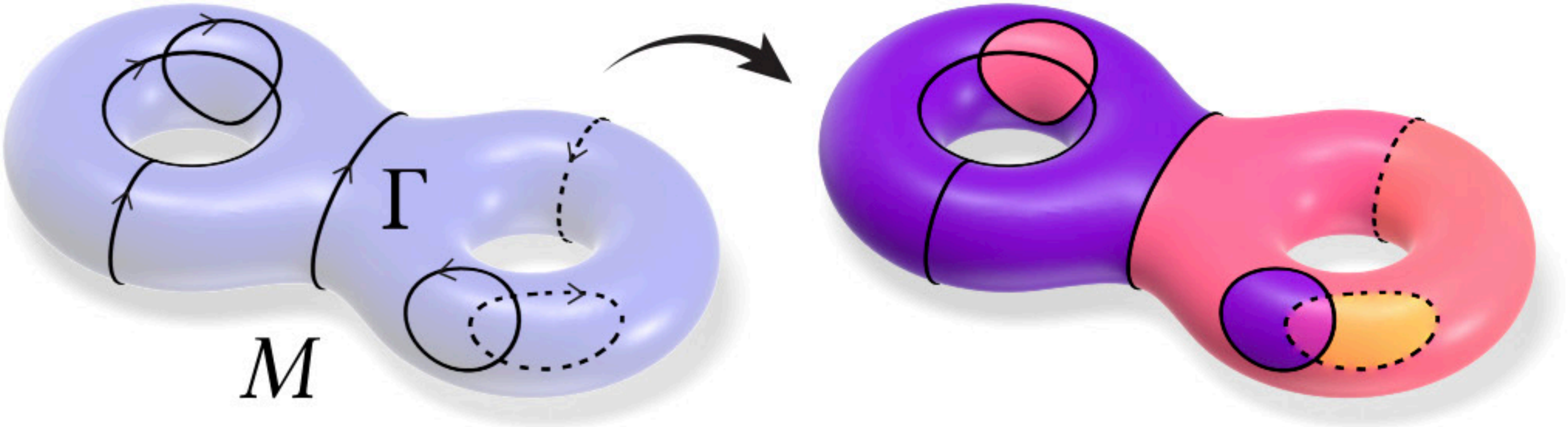

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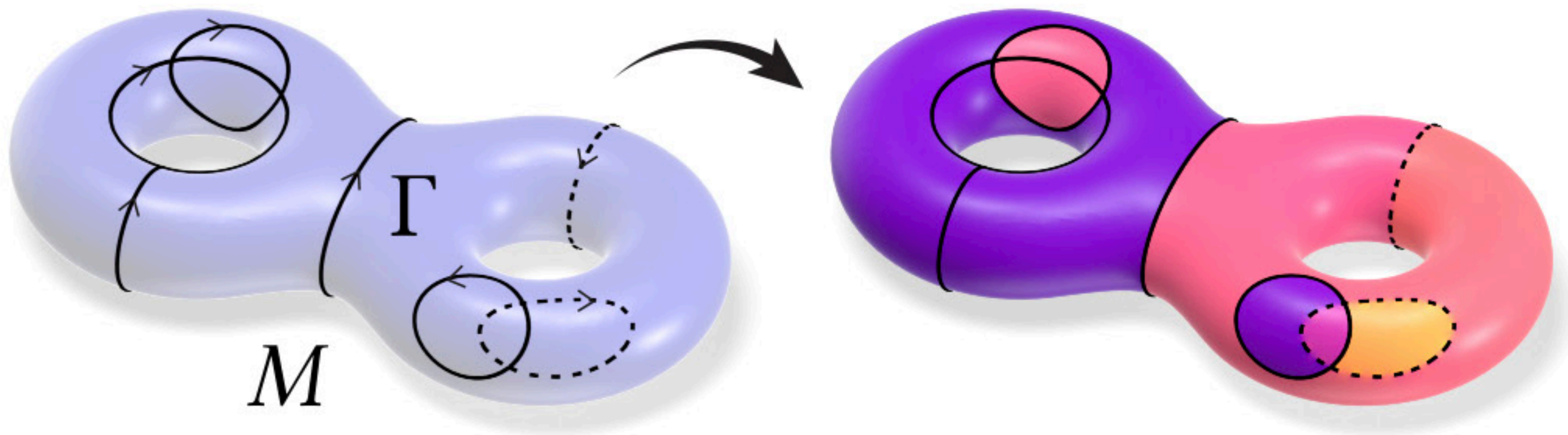

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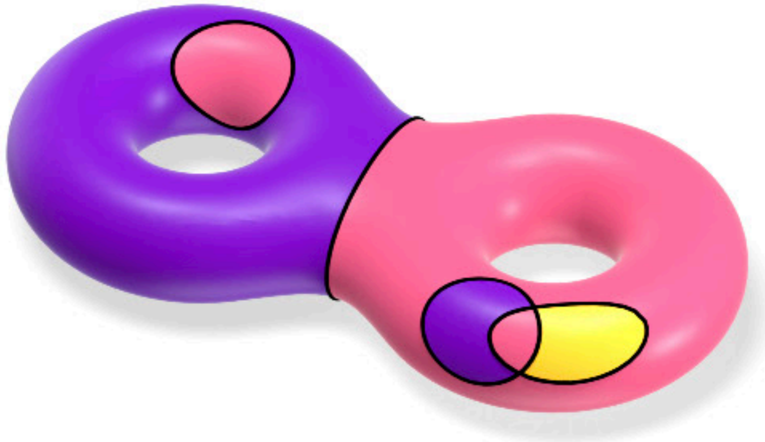
- bounding components that induce valid regions



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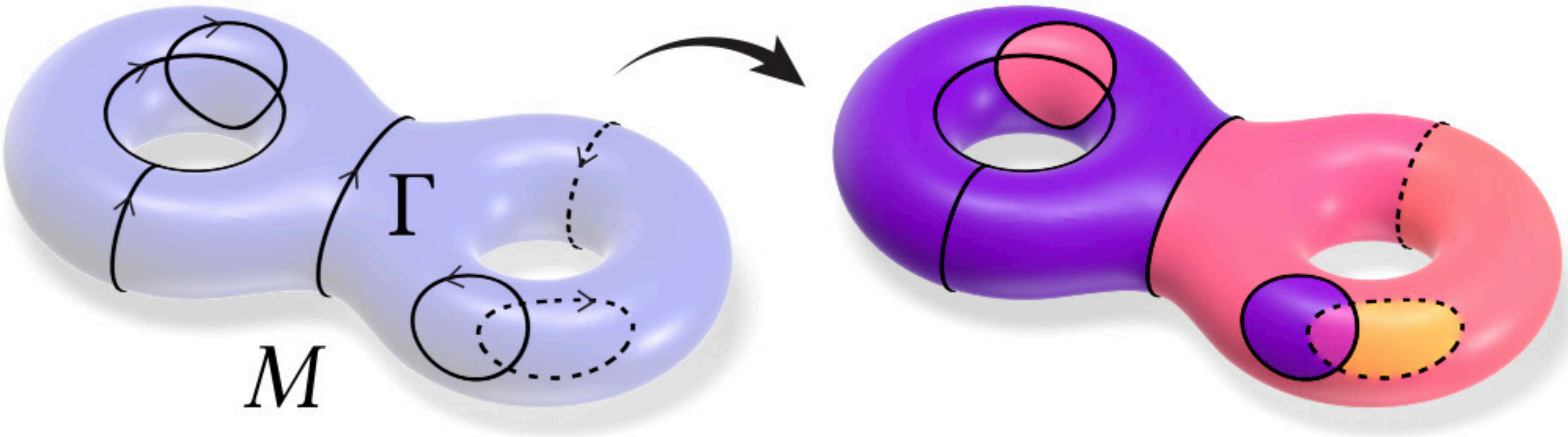

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A decomposition of Γ into:

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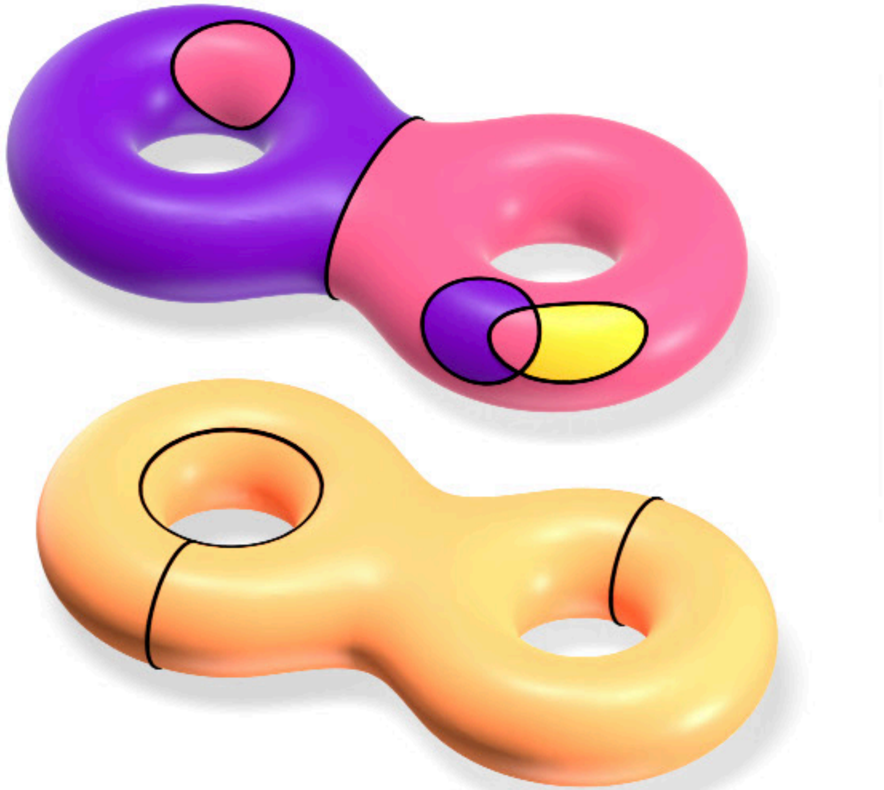
- nonbounding components.



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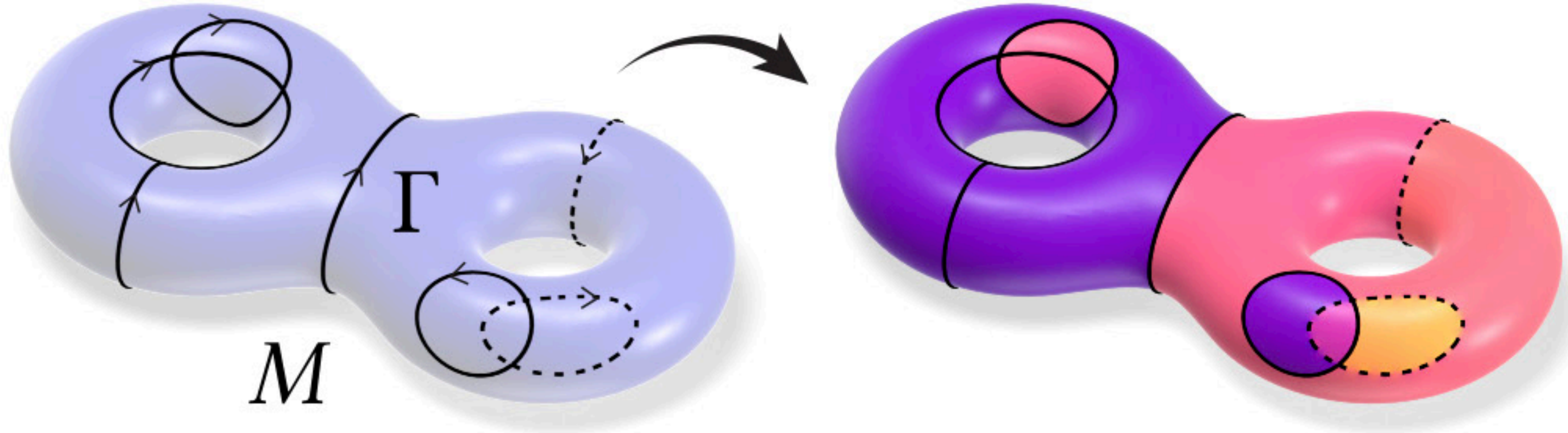

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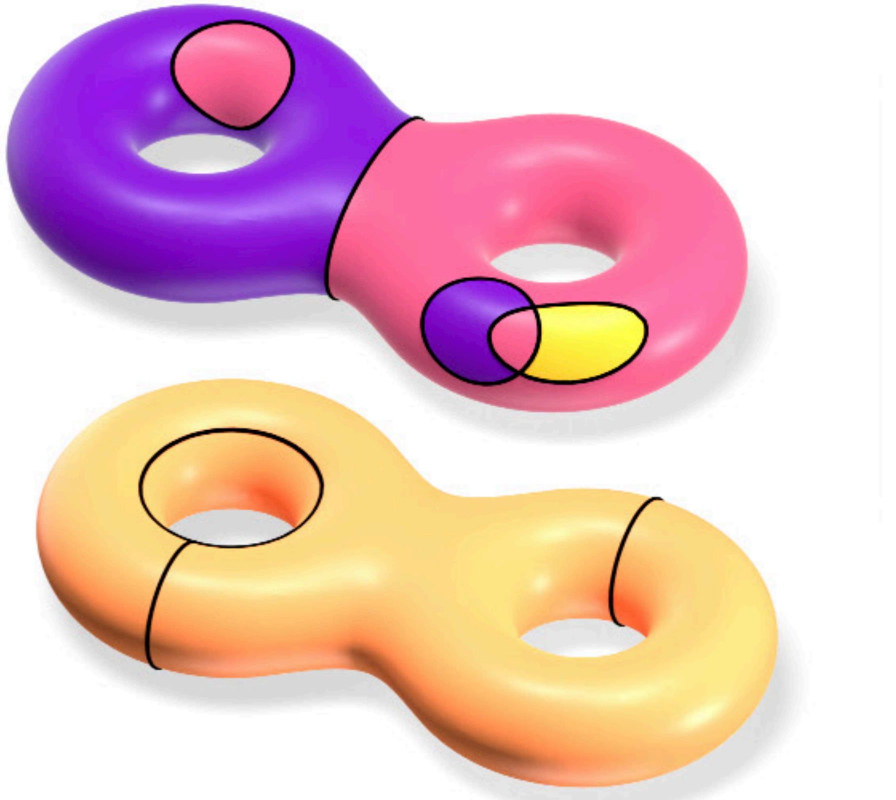
A closed, completed version of the input curve.



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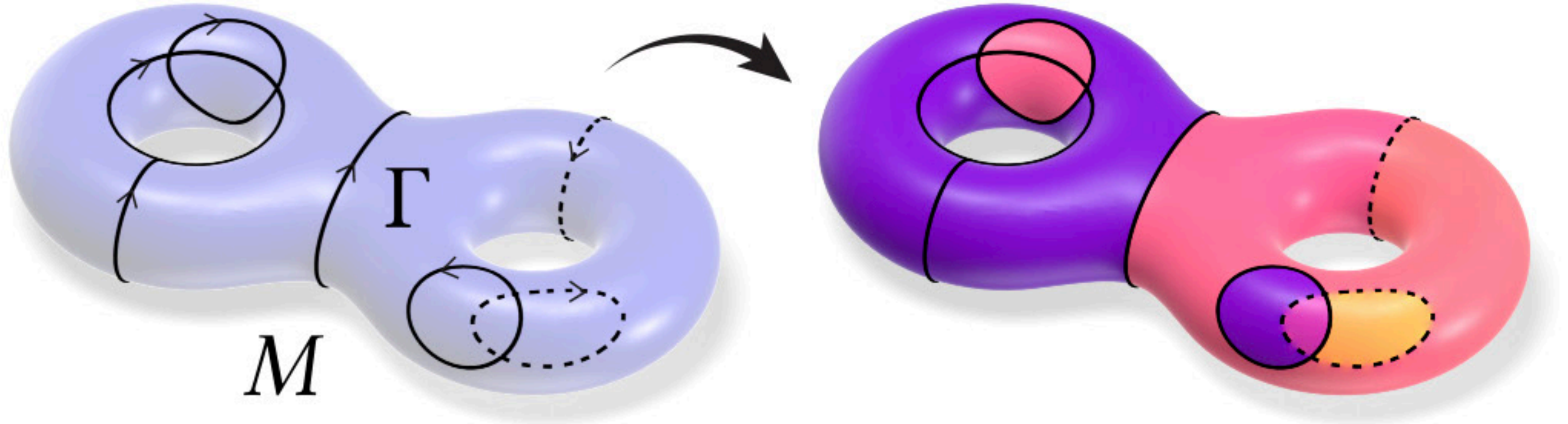
Γ is ***not*** partitioned into loops — no labels!



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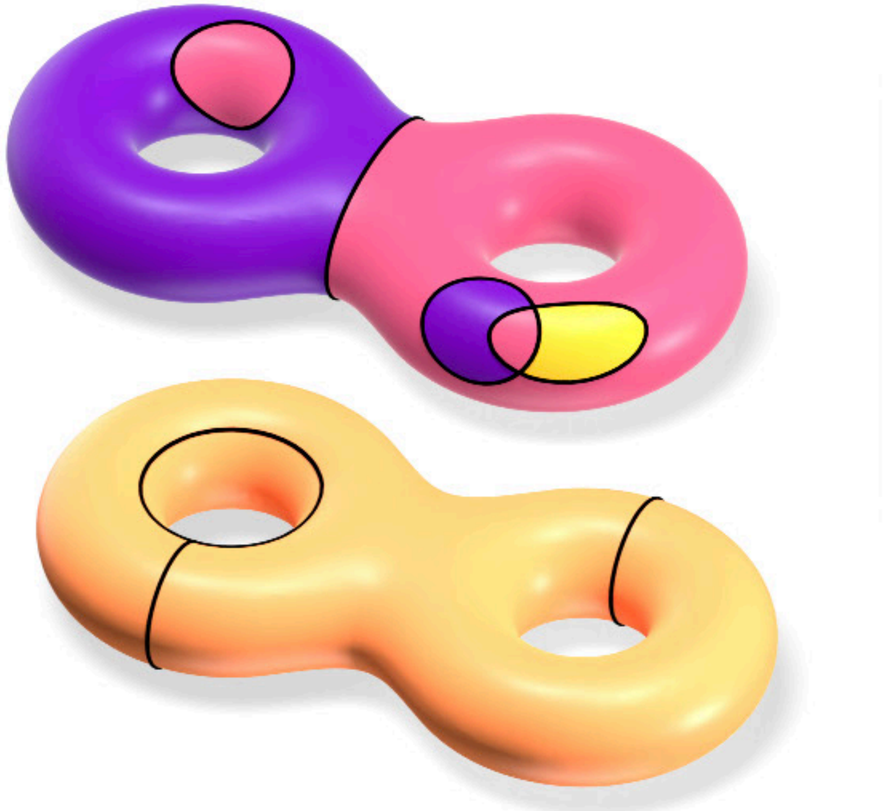
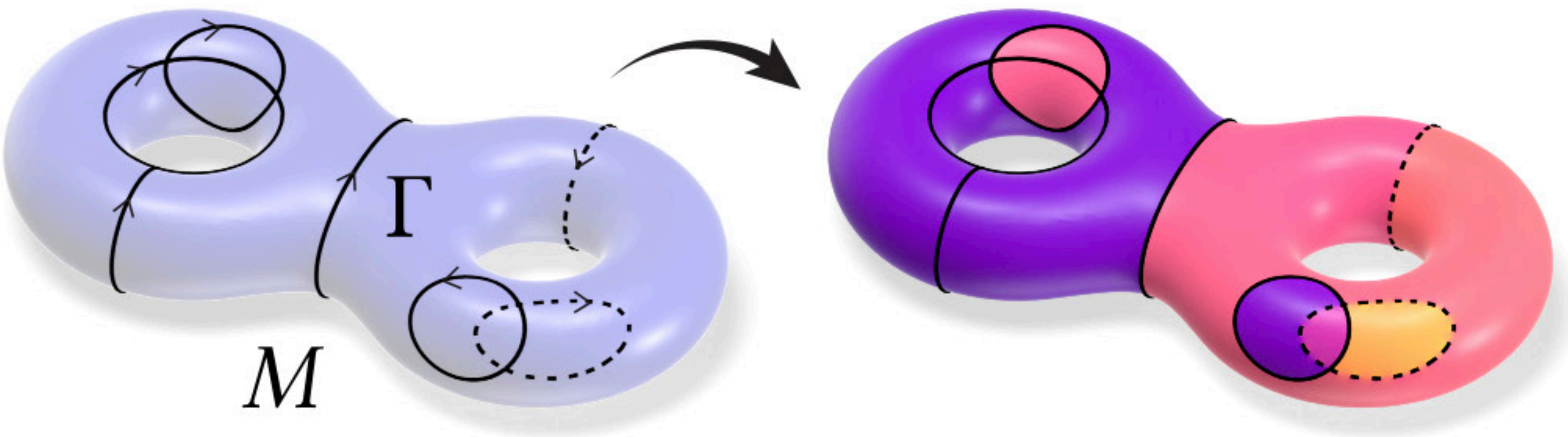
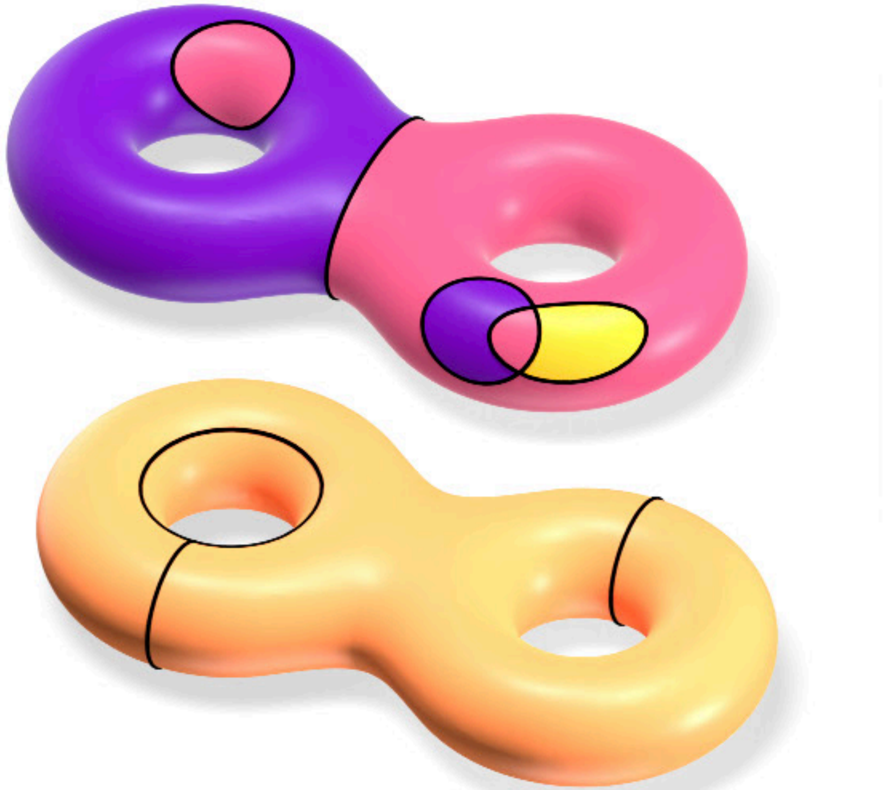


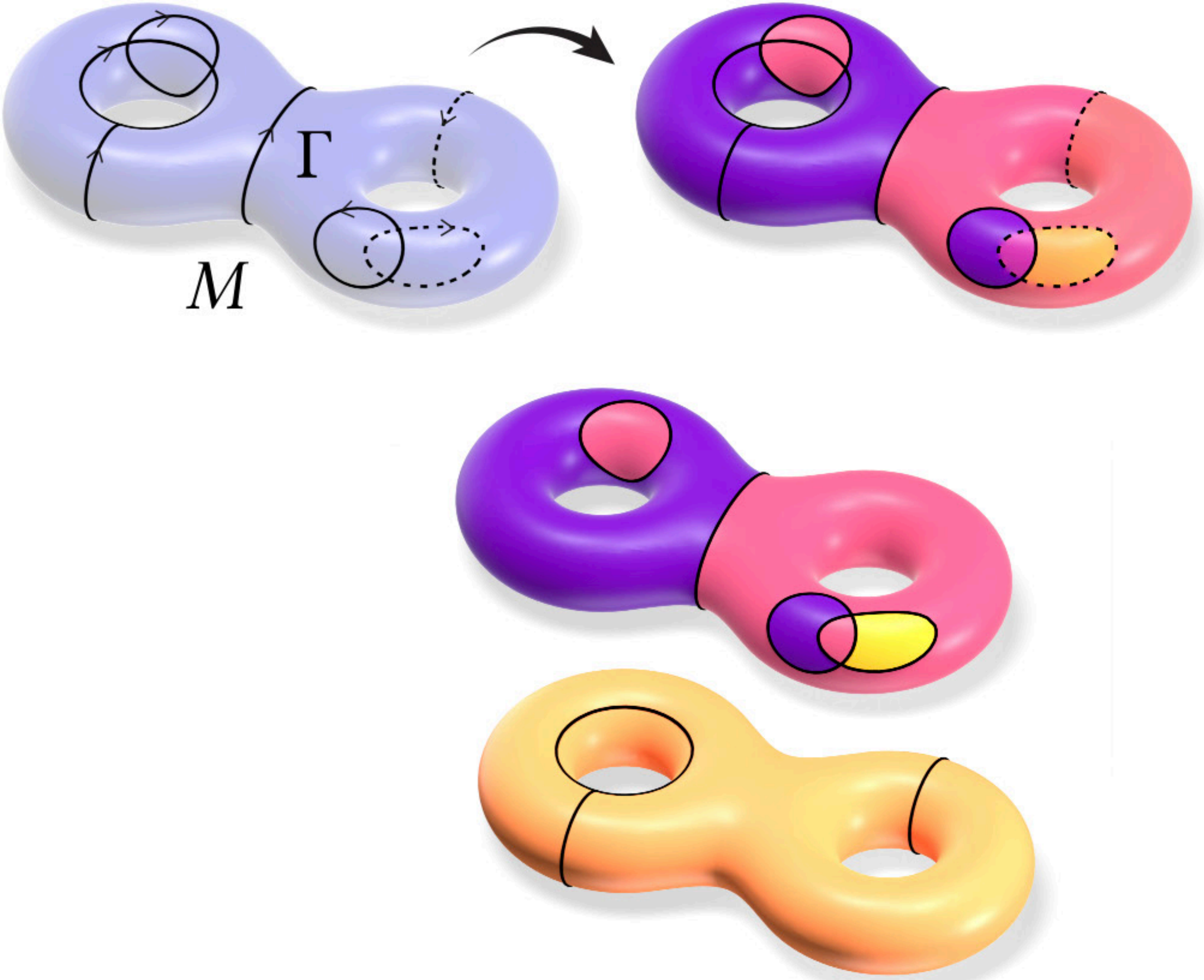
Key insights





Key insights

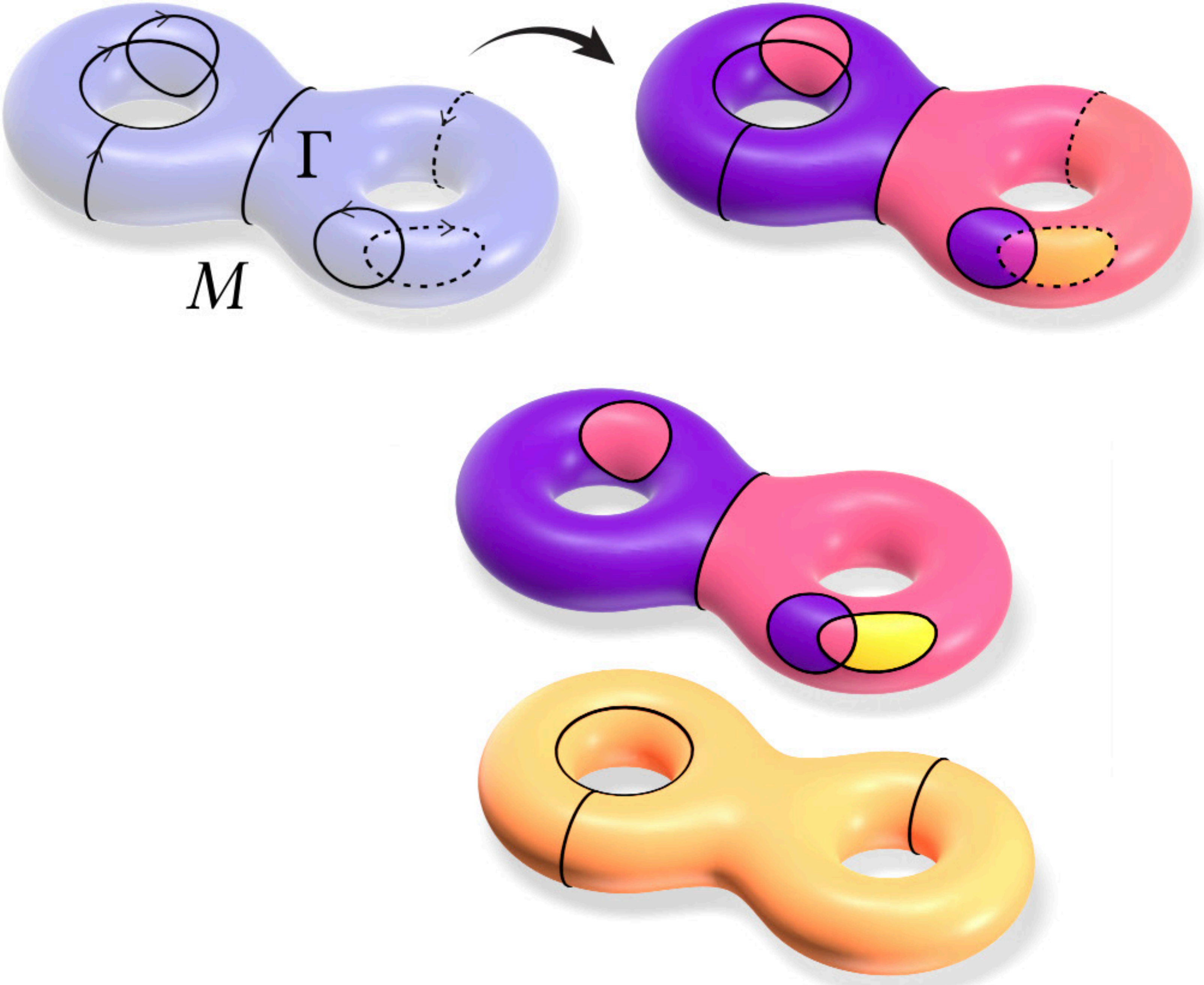
It's difficult to reason about the homology class of *broken* curves.



Key insights

It's difficult to reason about the homology class of *broken* curves.

Instead of processing curves directly, we process functions *dual* to curves using *de Rham cohomology*.



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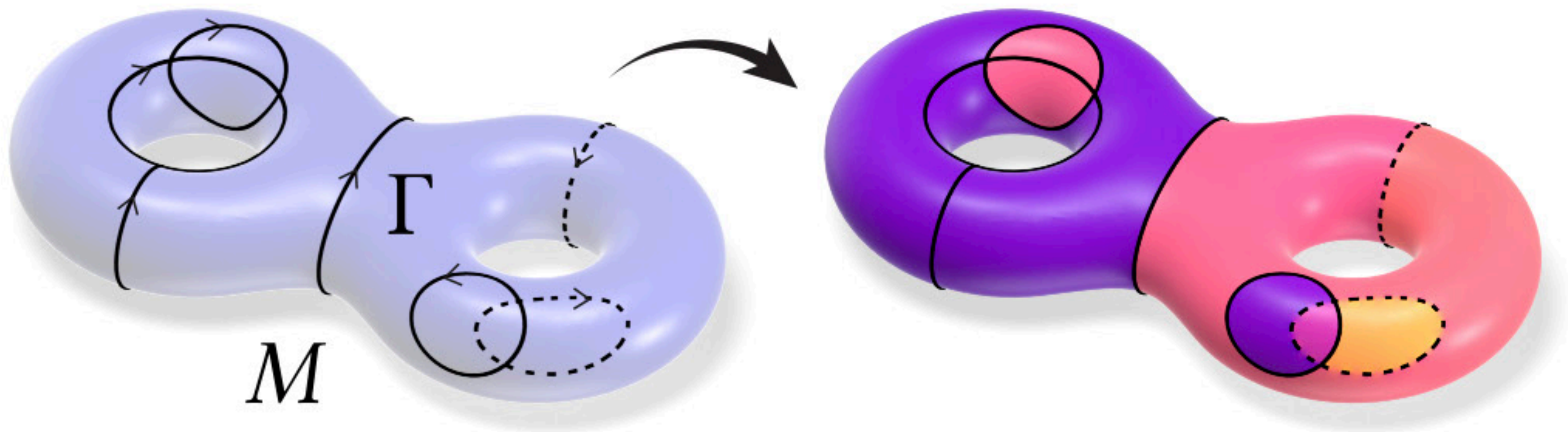
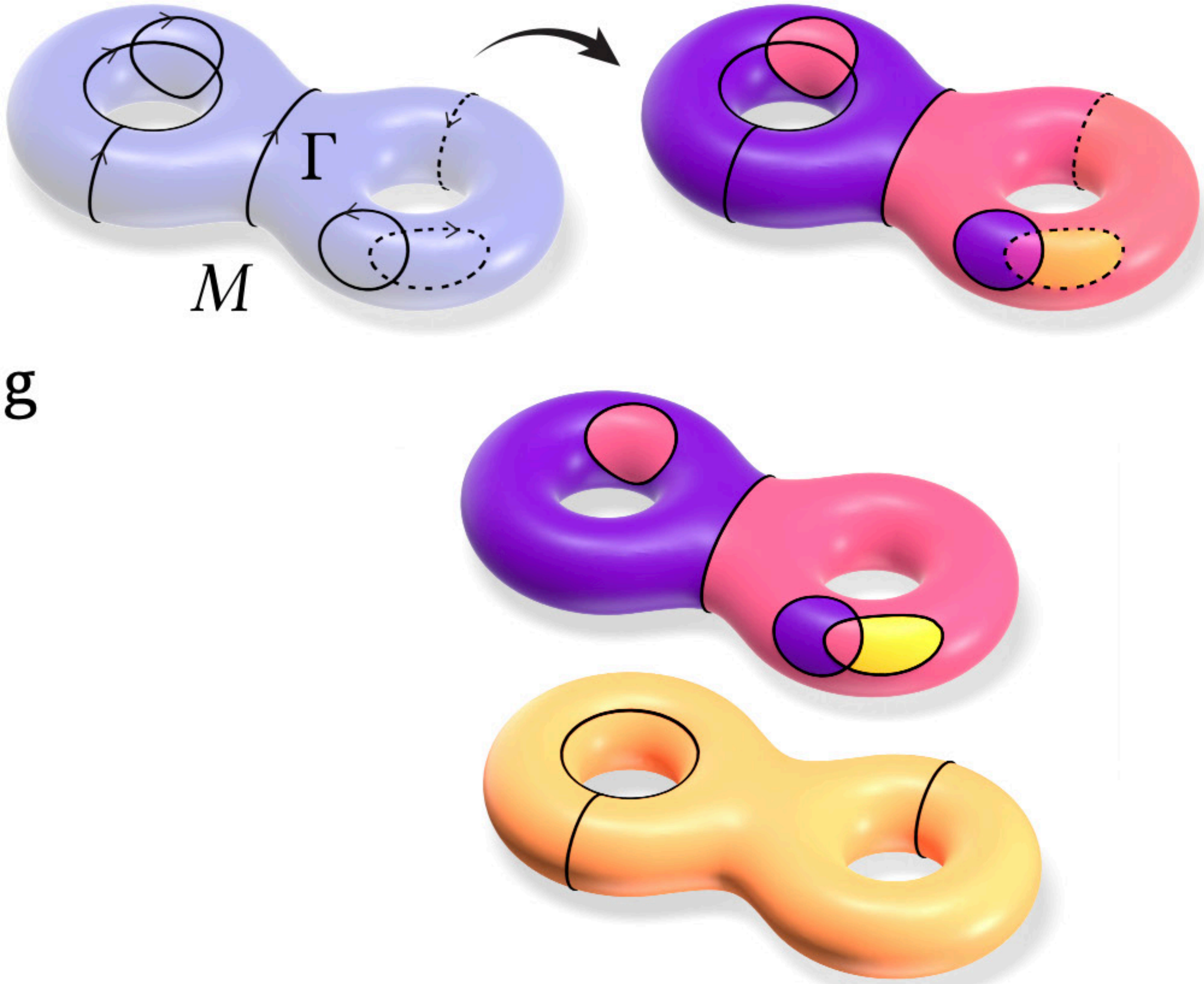


Instead of processing curves directly, we process functions *dual* to curves using *de Rham cohomology.*

We map from functions back to curves, yielding final output.

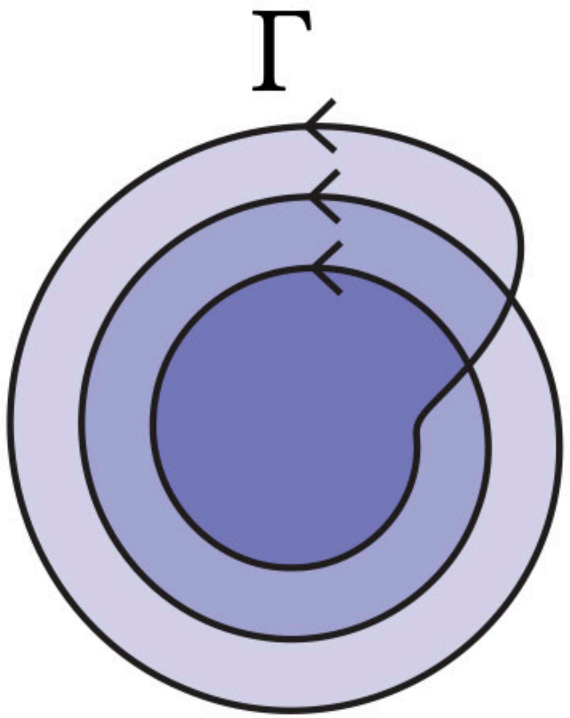
Talk outline



- Algorithm in the smooth setting
- Discretization
- Evaluation & Results

Smooth formulation

Ordinary winding number: a piecewise constant function



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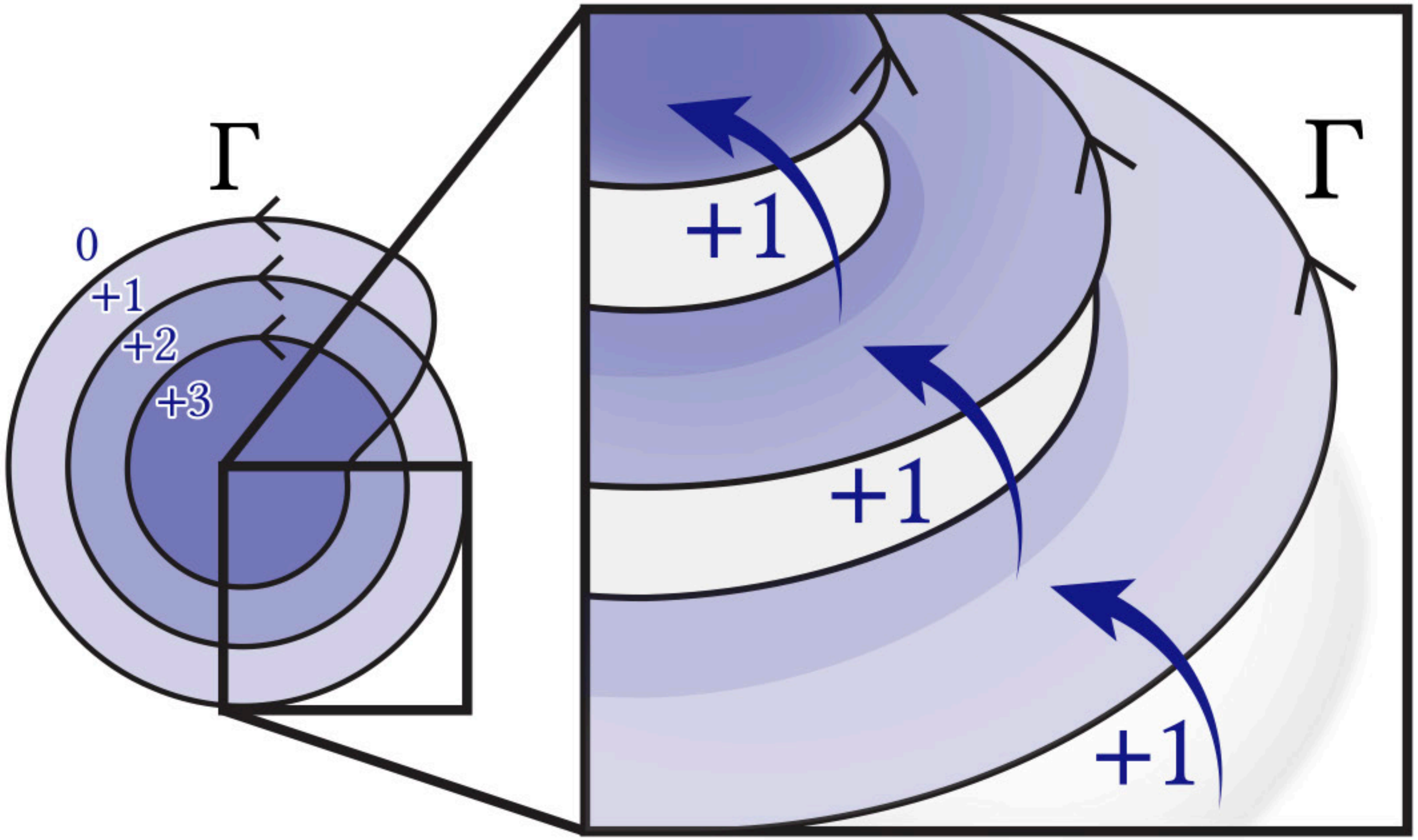

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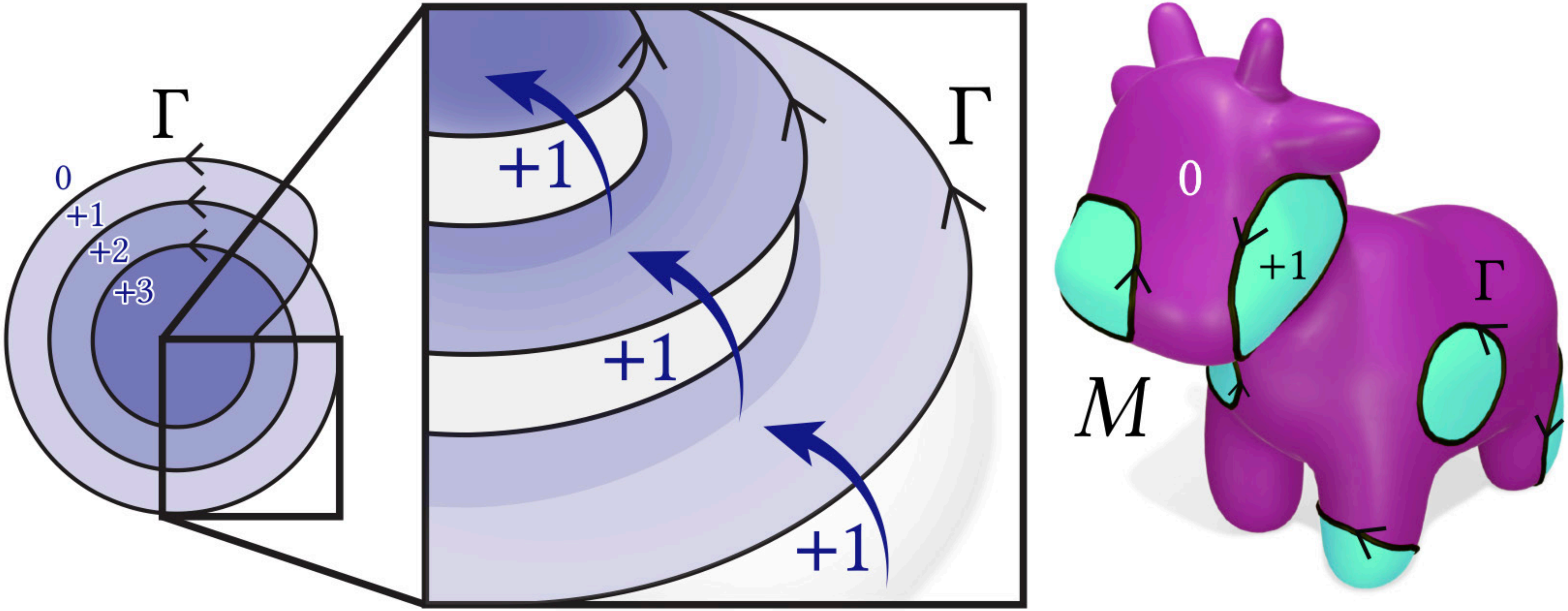
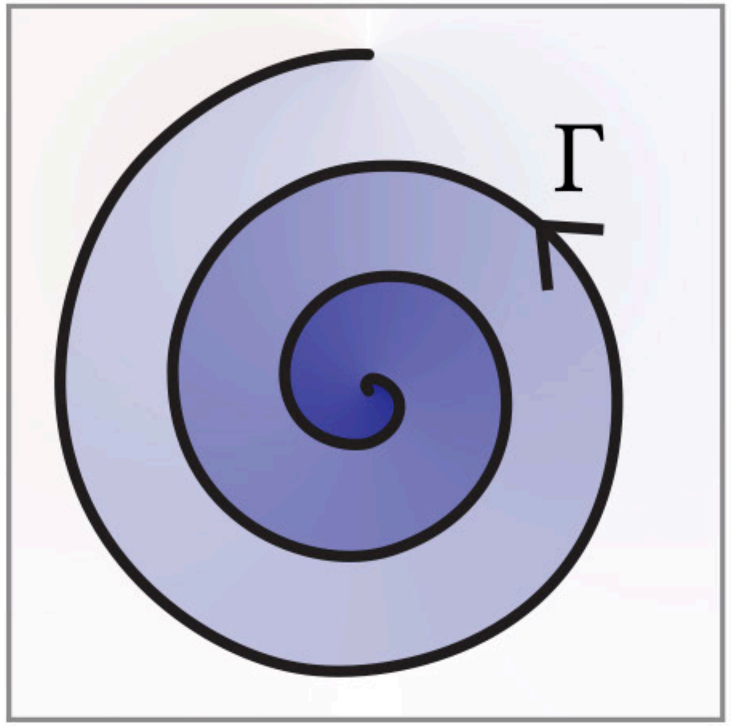


Winding number as a jump harmonic function



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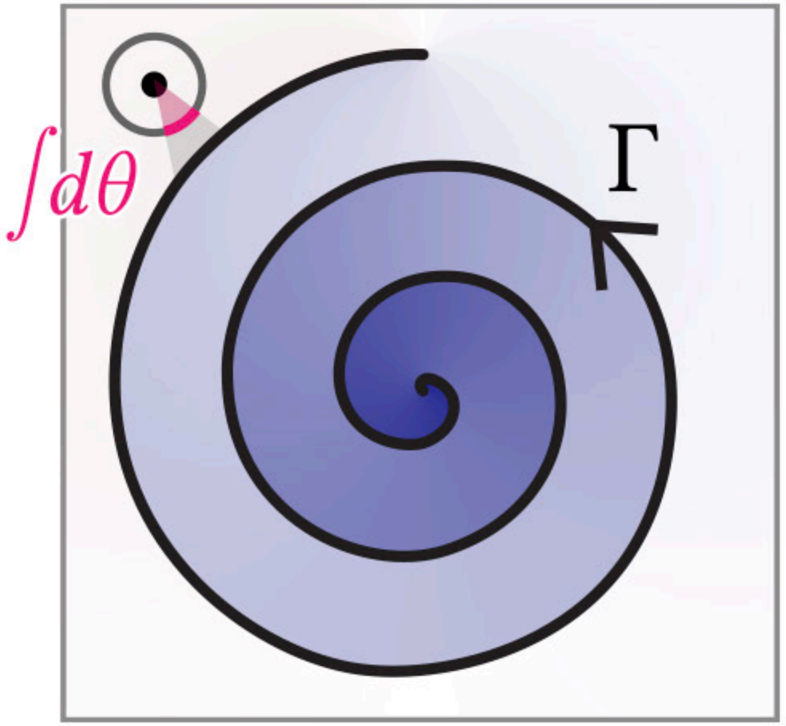

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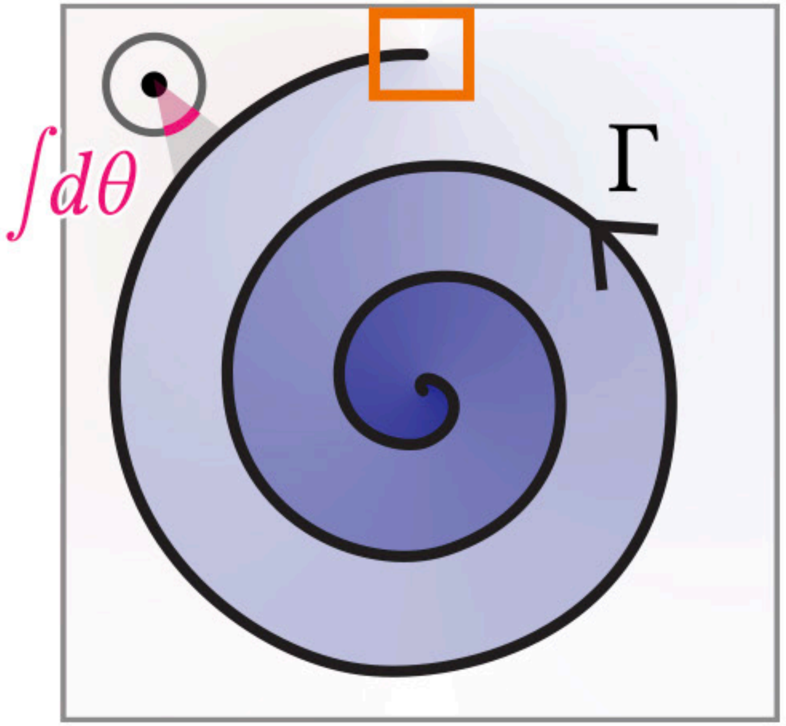
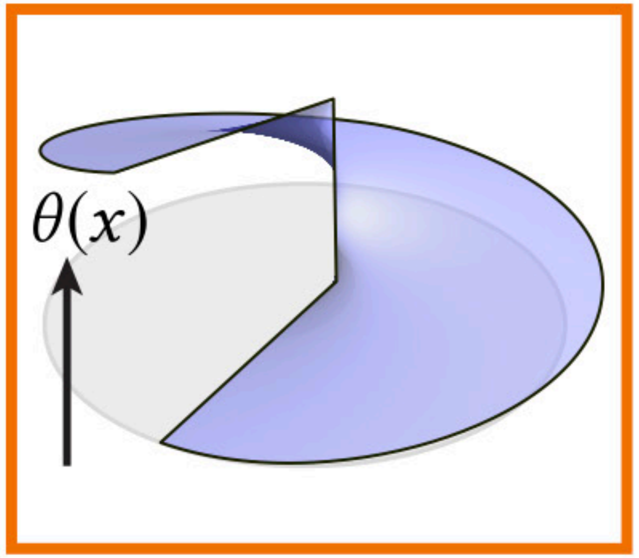
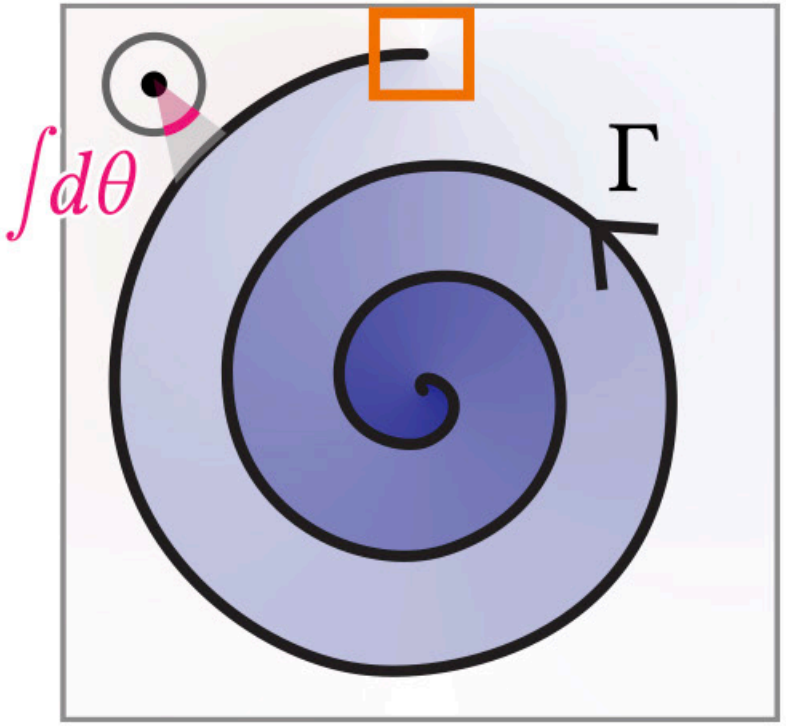
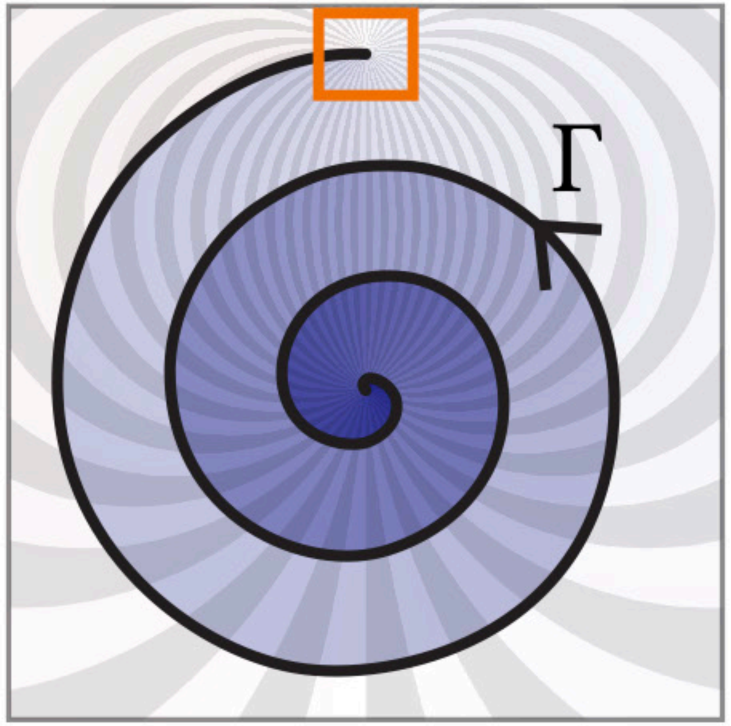
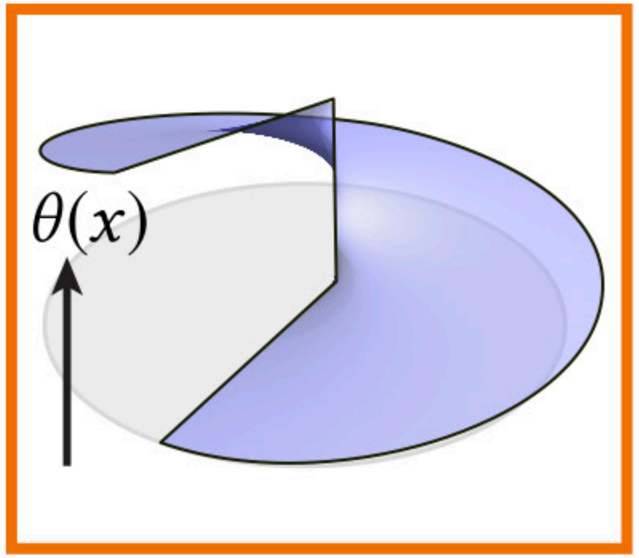


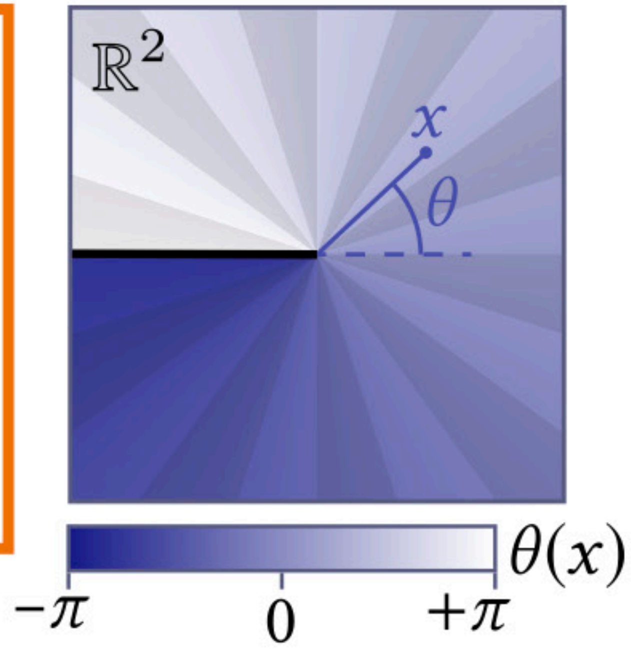


Winding number as an angle-valued function









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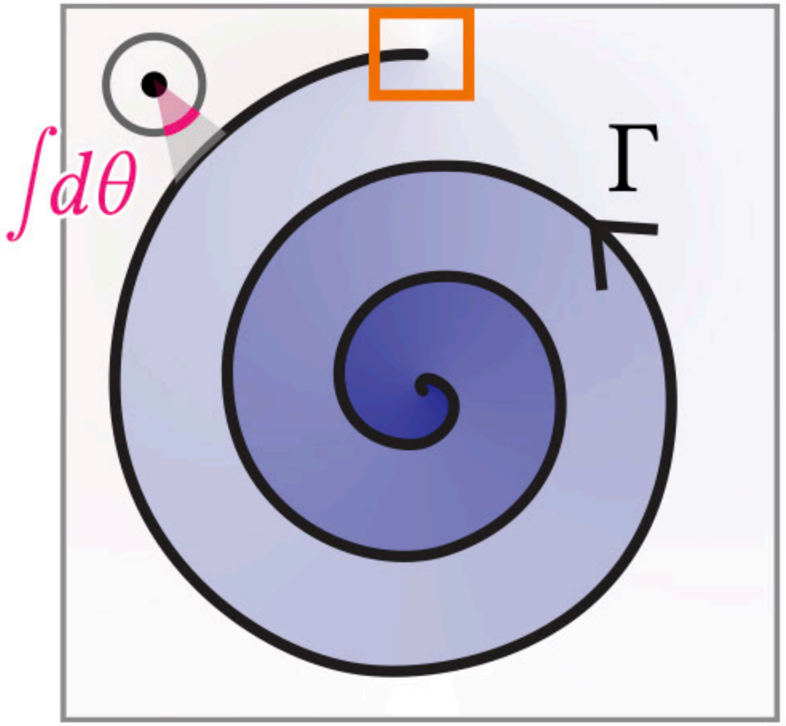

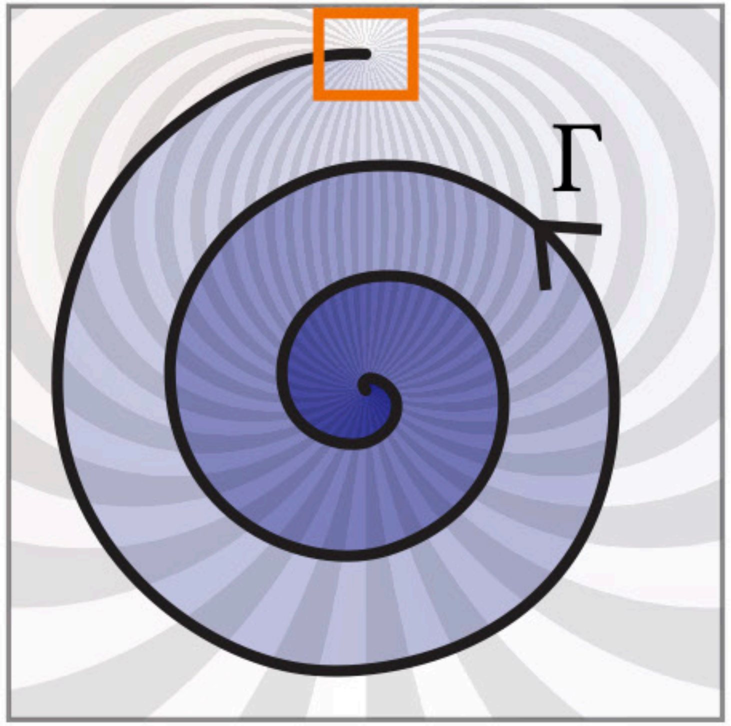

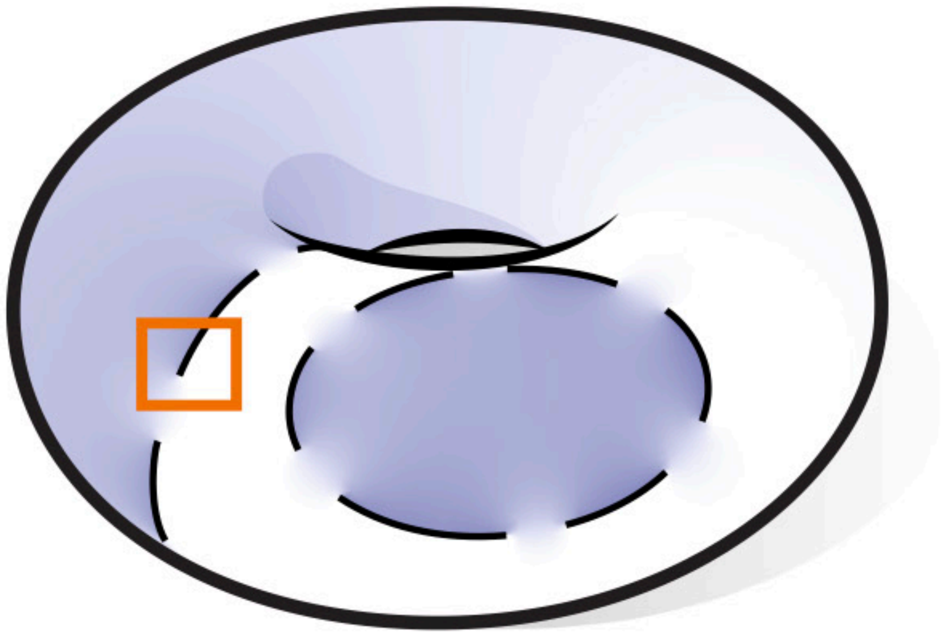

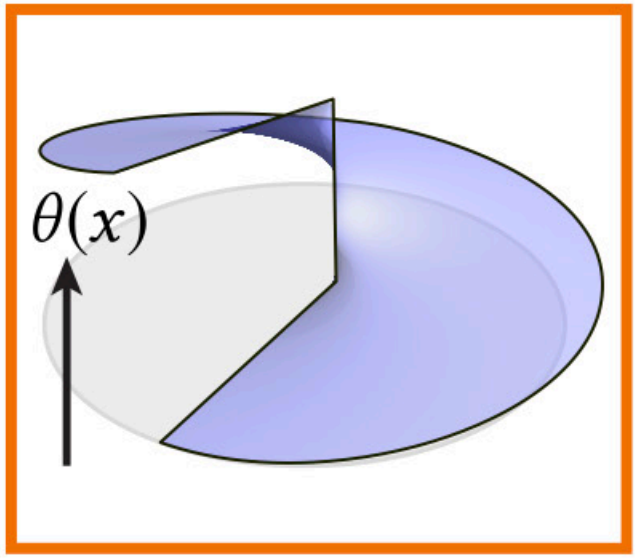

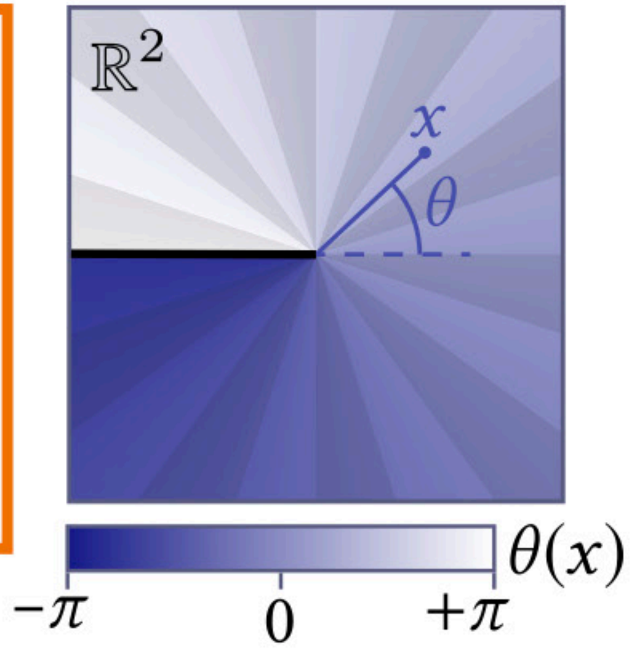

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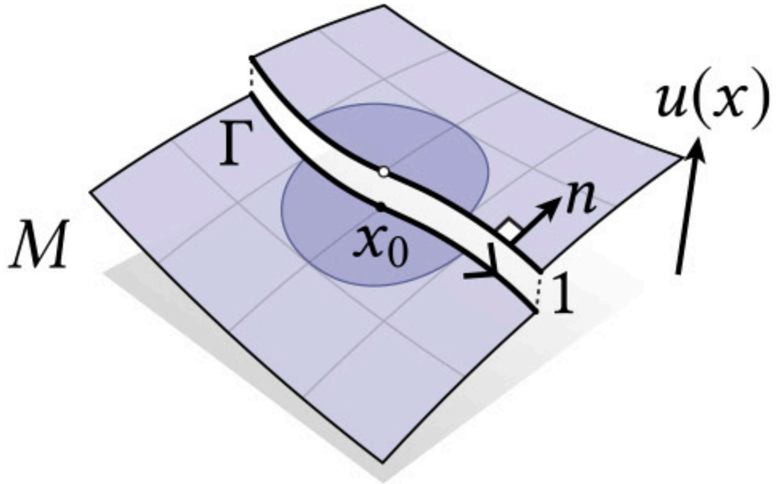

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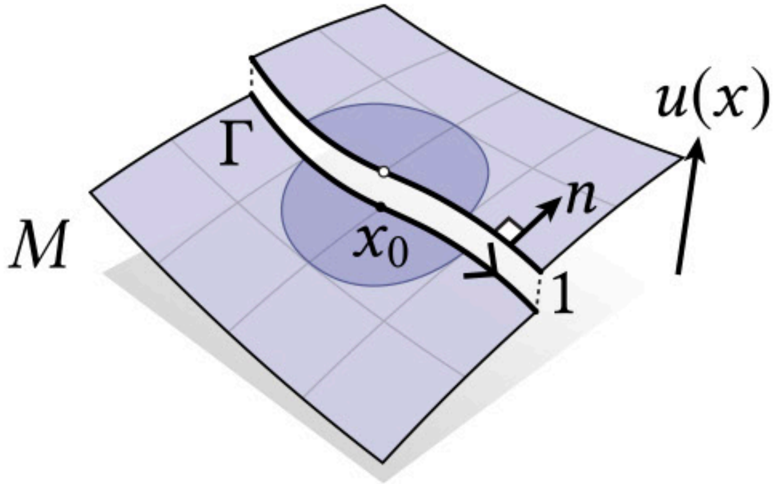

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$$
\begin{aligned}
\Delta u &= 0, && \text{on } M \setminus \Gamma, \\
u^+ - u^- &= 1^*, && \text{on } \Gamma, \\
\partial u^+ / \partial n &= \partial u^- / \partial n, && \text{on } \Gamma.
\end{aligned}
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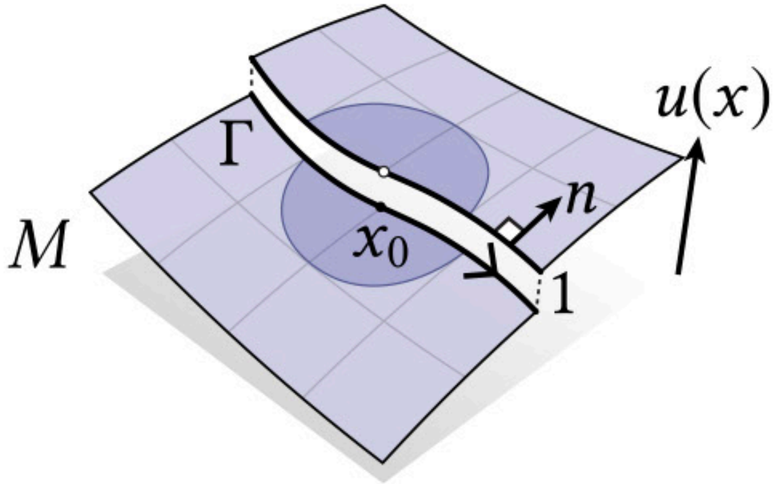

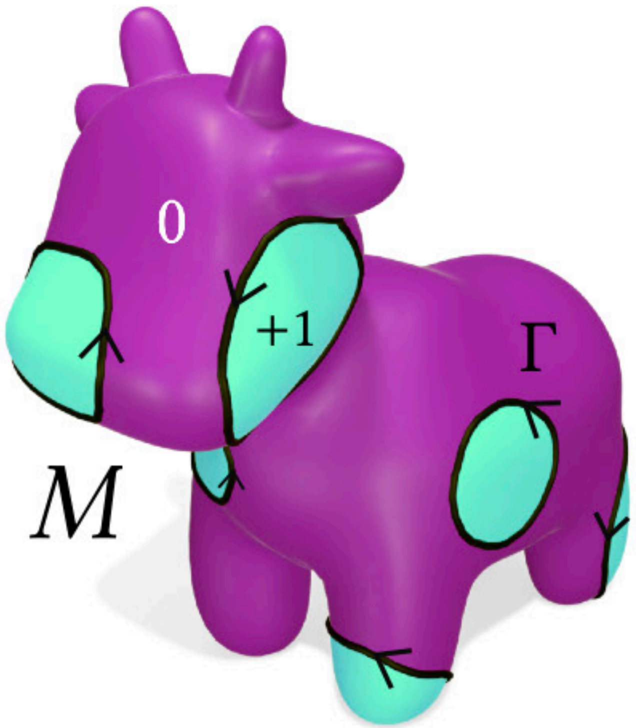

*In general, $u^+(x) - u^-(x) =$ number of times Γ covers x.

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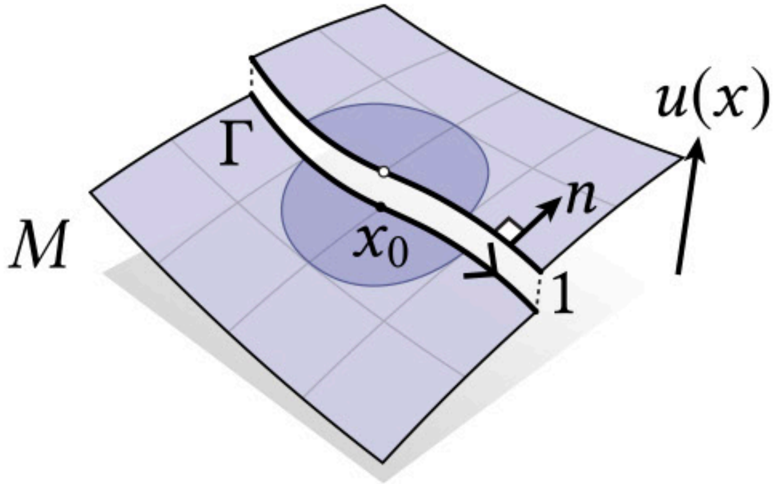
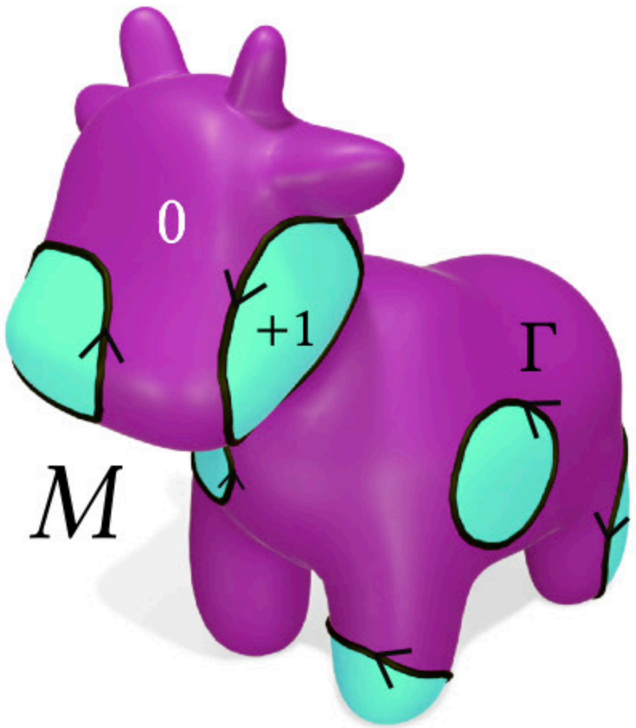


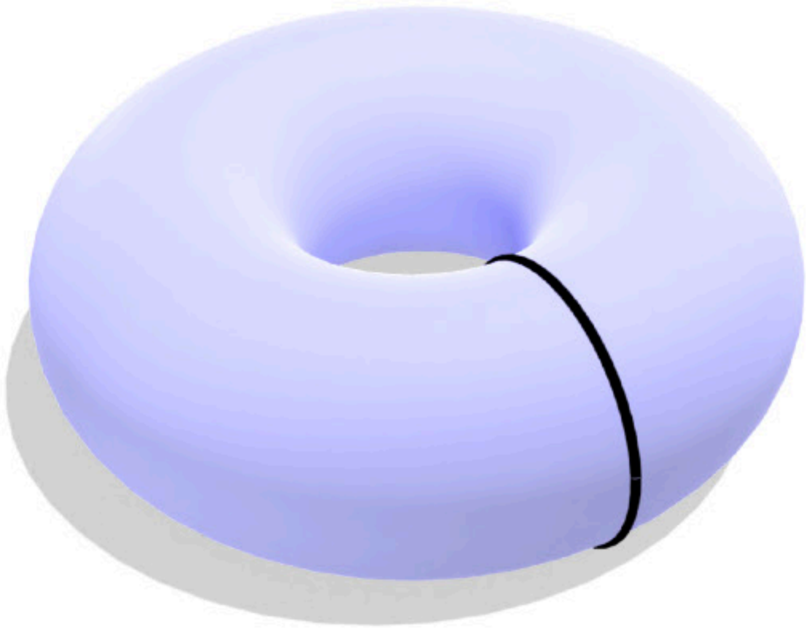


If M is simply-connected, and all curves are closed, then we're done!

*In general, $u^+(x) - u^-(x)$ = number of times Γ covers x.

Nonbounding loops complicate region labeling

input



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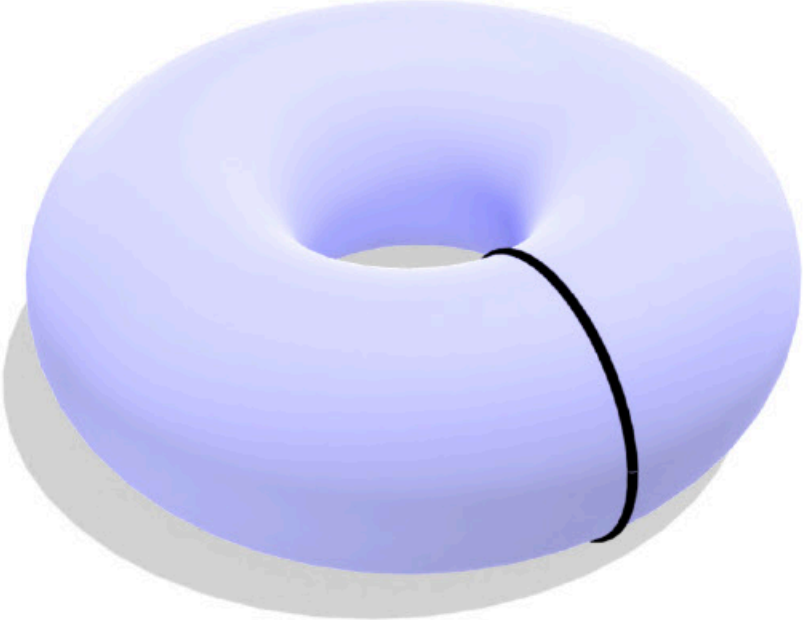

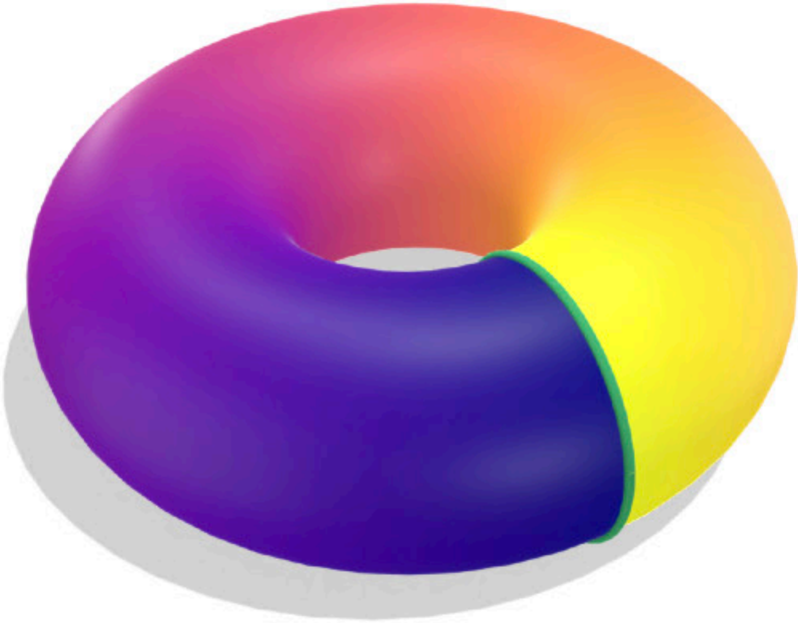

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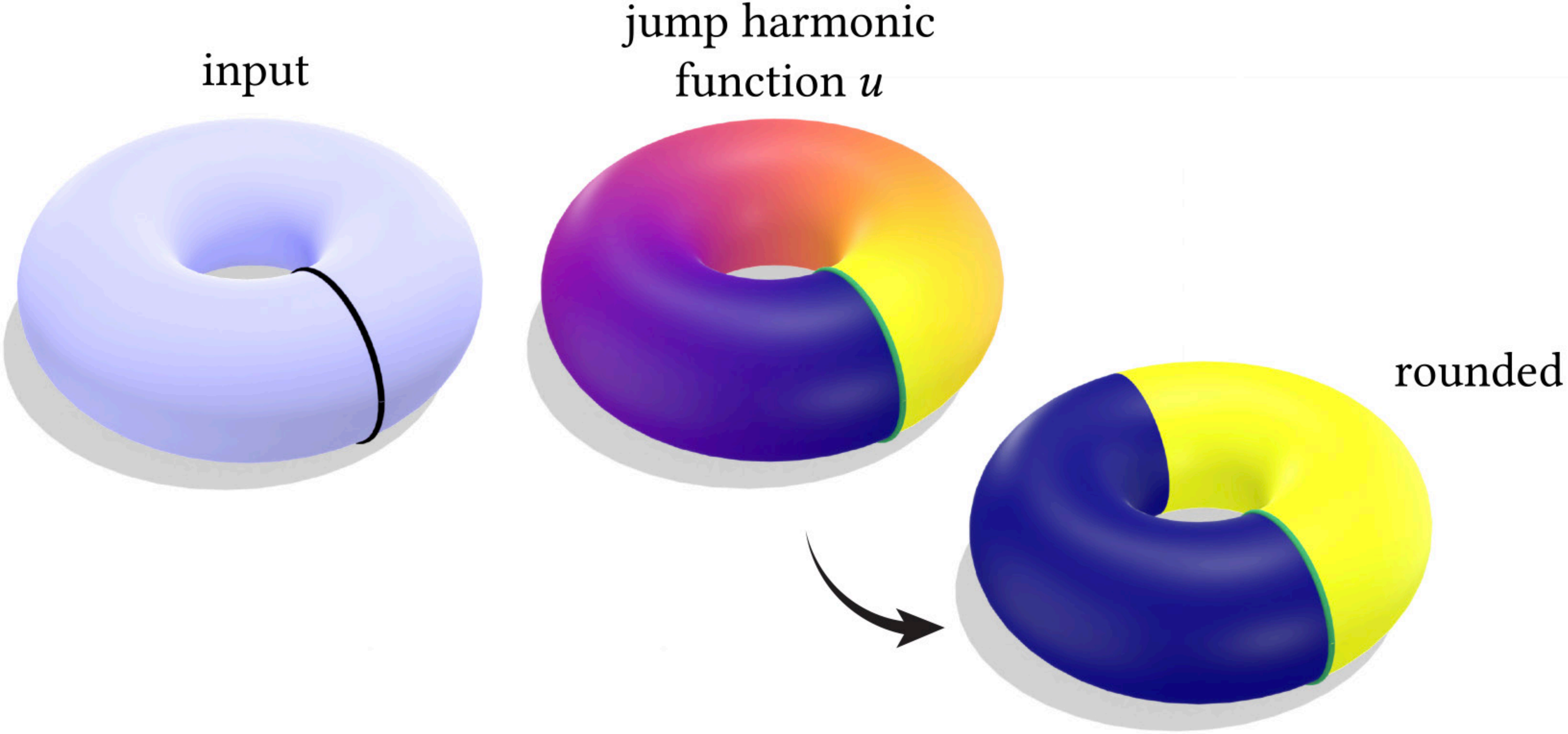

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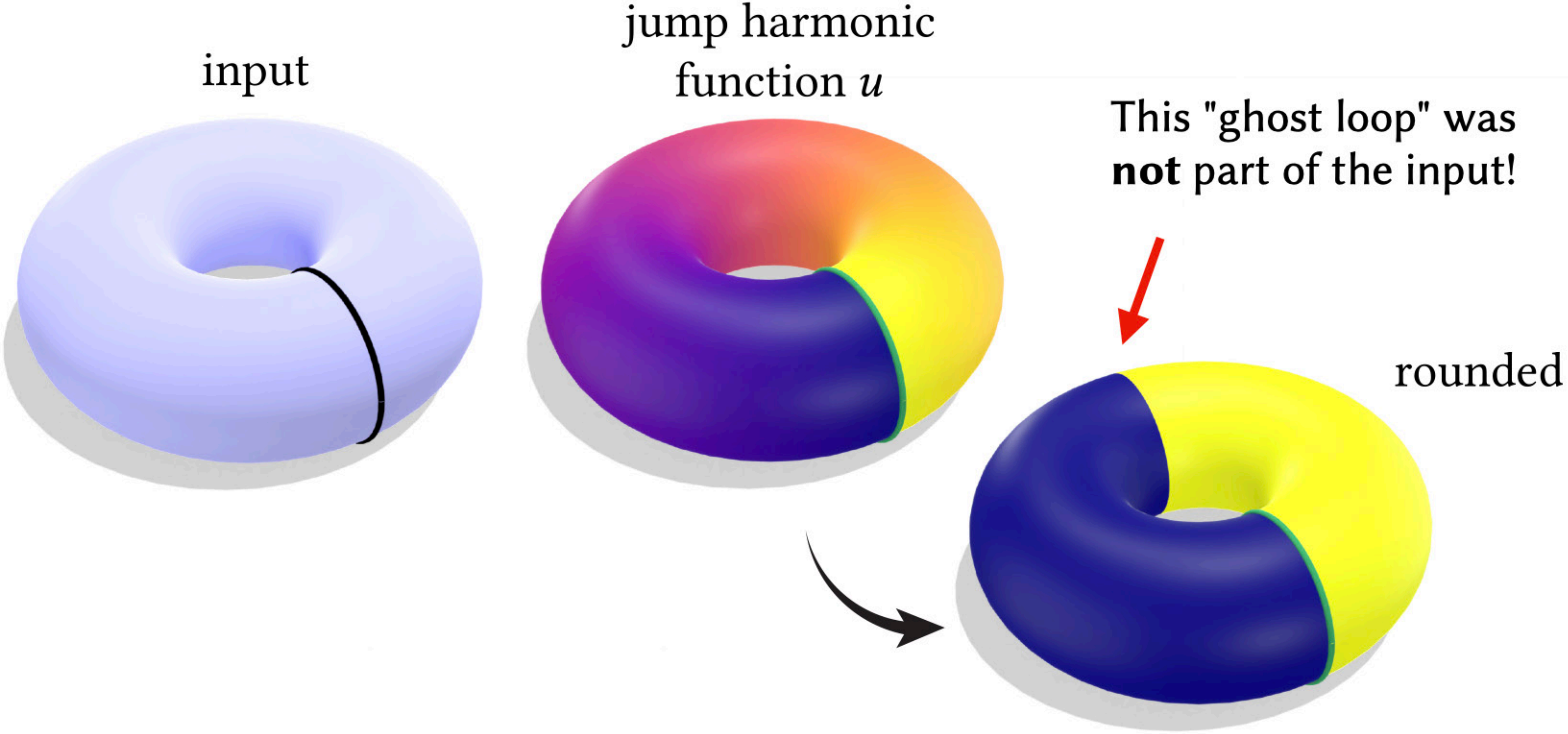

What are nonbounding loops?

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The *first homology group* $H_1(M) = \ker(\partial_1) \setminus \operatorname{im}(\partial_2)$ tells us about (closed) curves that are not boundaries of regions.

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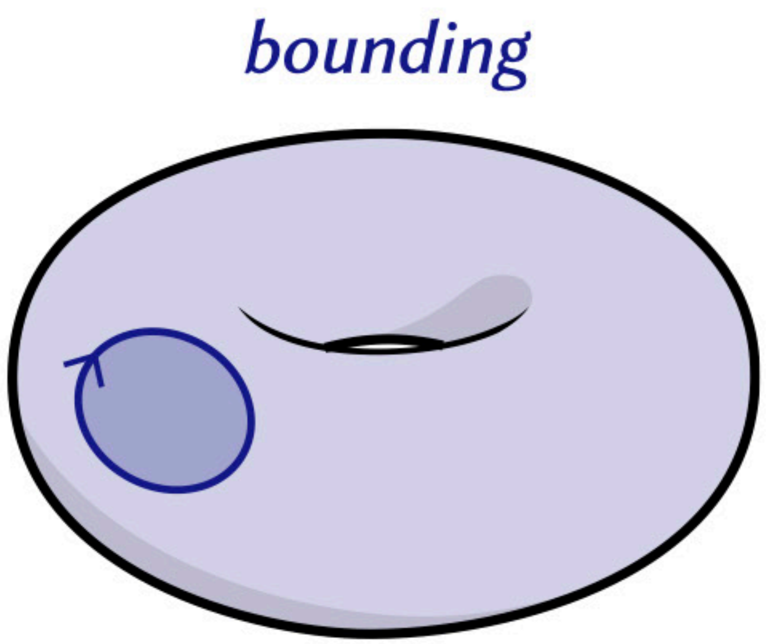

For clarity,

bounding := *nullhomologous*

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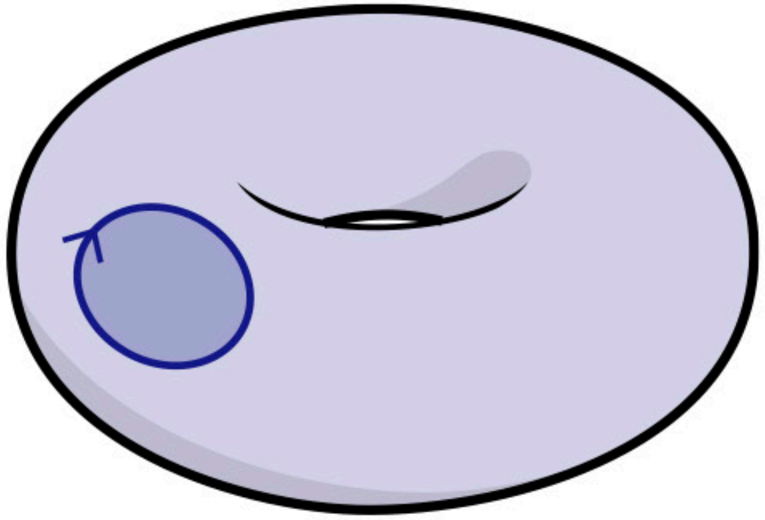
For clarity,

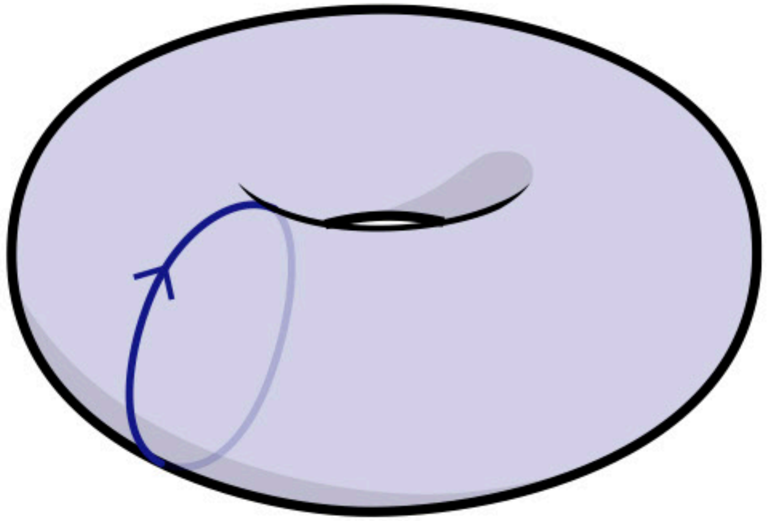
bounding := *nullhomologous*

nonbounding := *non-nullhomologous*

bounding



nonbounding



How does homology apply to broken curves?

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It’s difficult to reason about curves directly.

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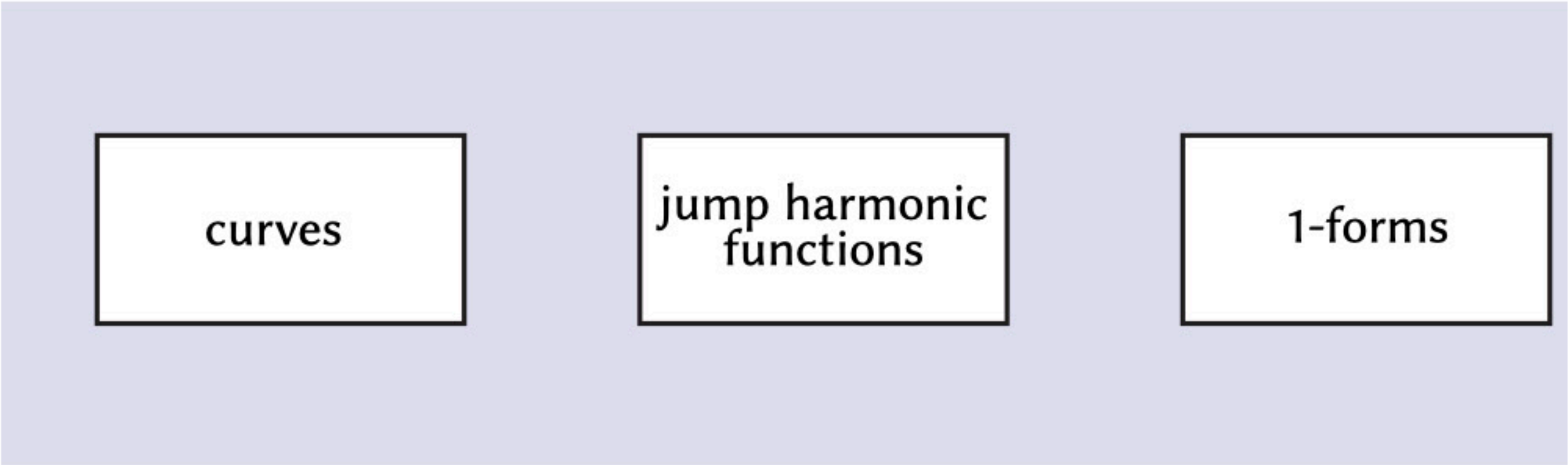
Instead of processing curves directly, we process functions *dual* to curves using *de Rham cohomology.*

A preview of the journey ahead



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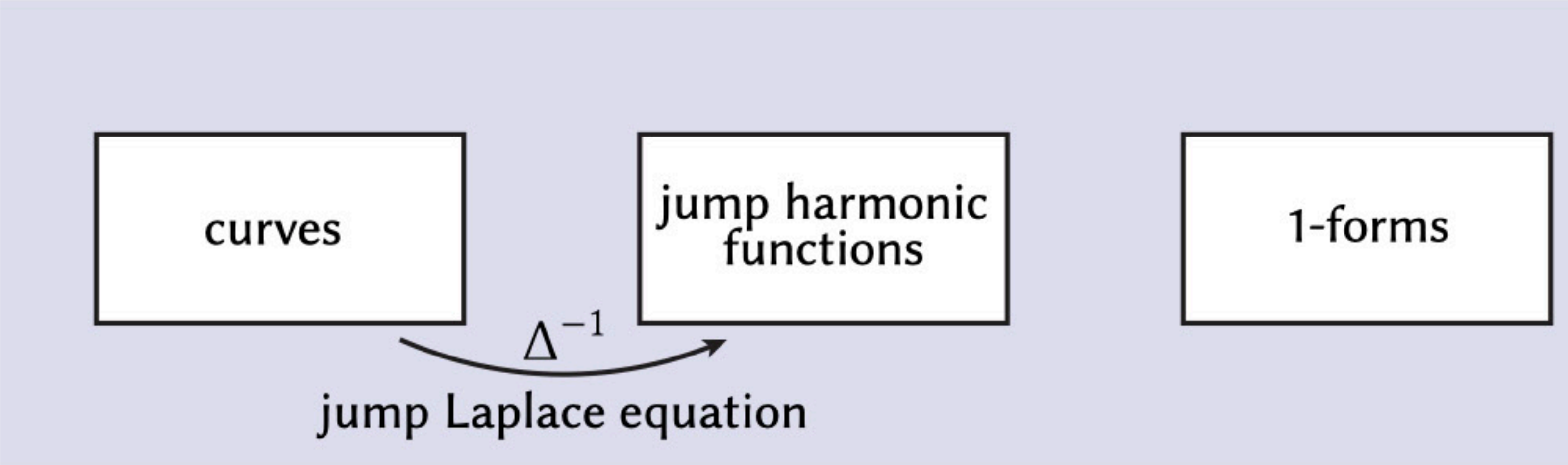

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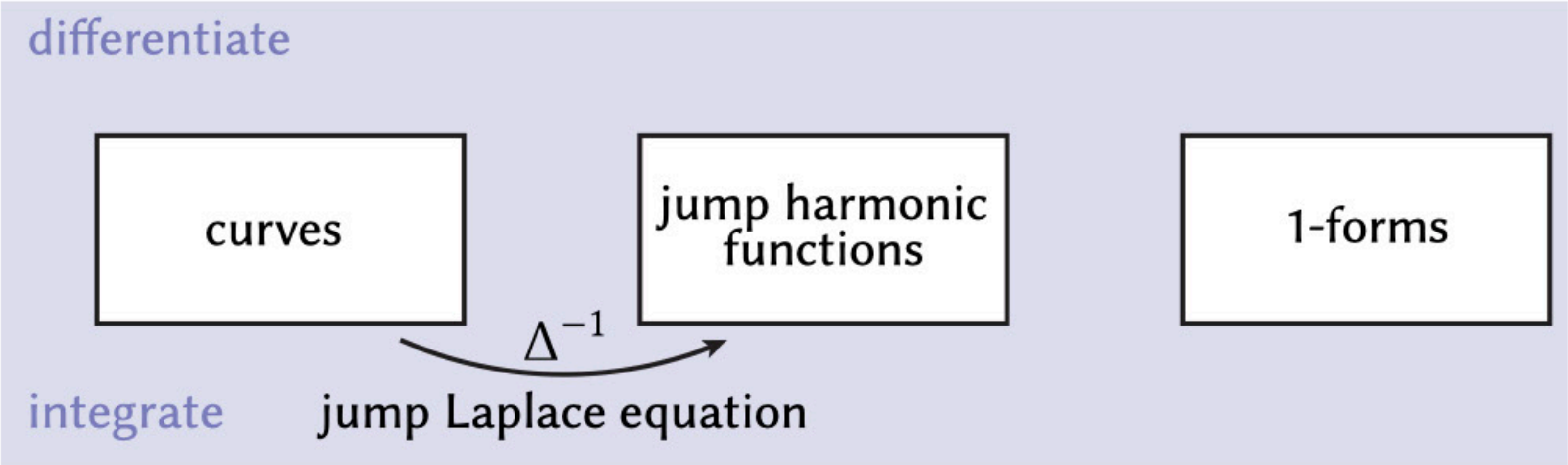

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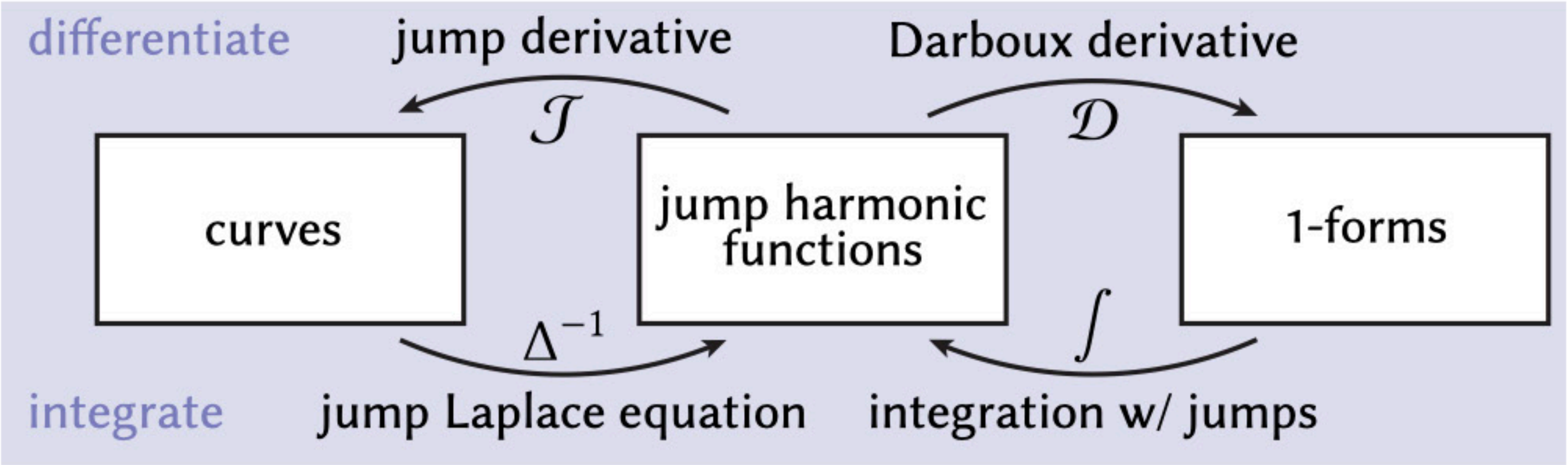
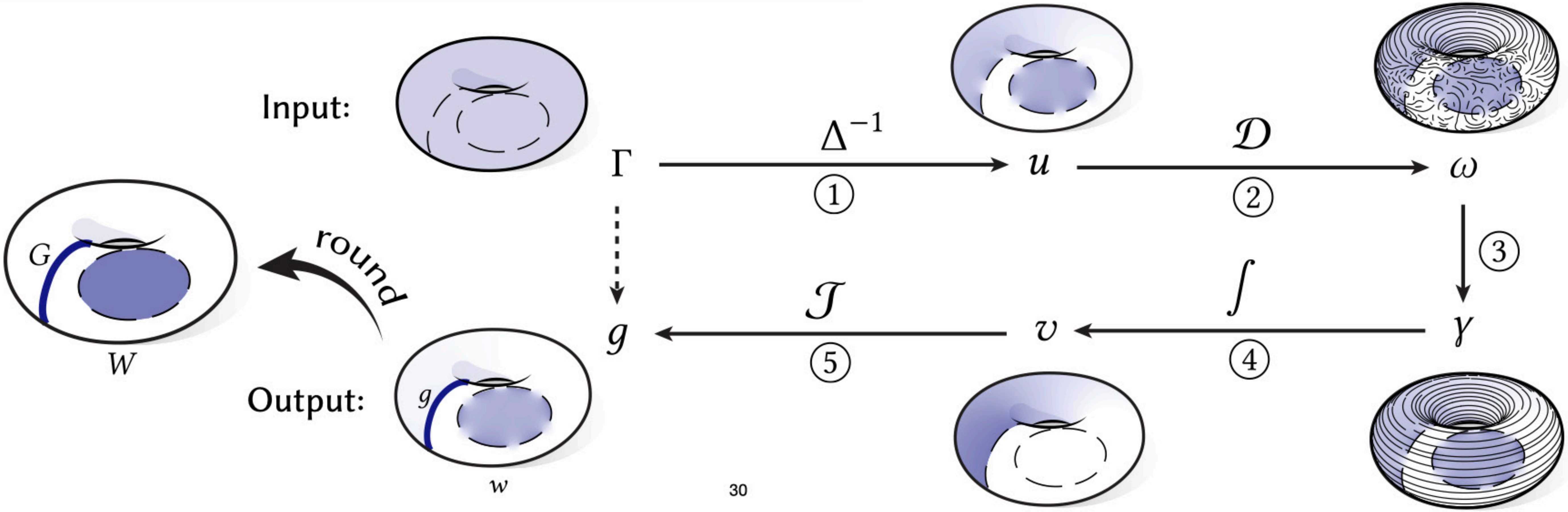
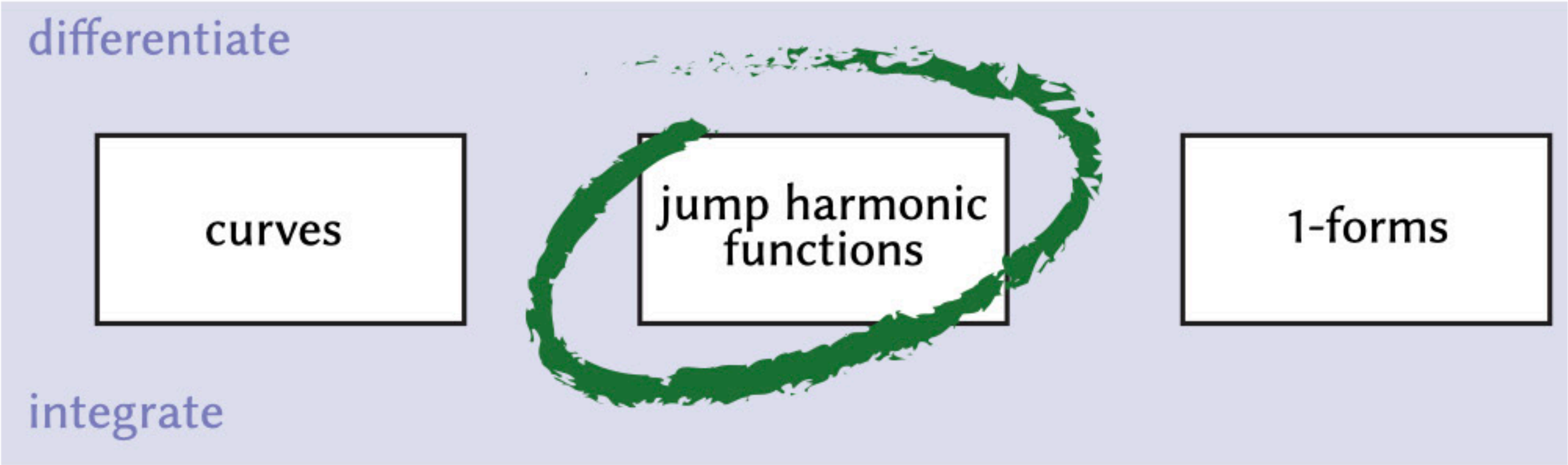




Jump harmonic functions

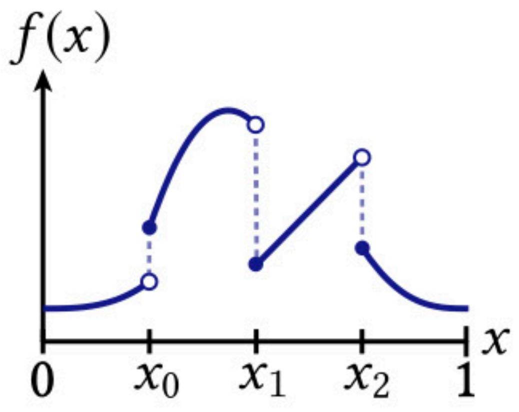


First let's talk about differentiating & integrating jump harmonic functions.

Derivatives of discontinuous functions

Derivatives of discontinuous functions

Consider a periodic 1D function $f(x)$ on [0,1]:



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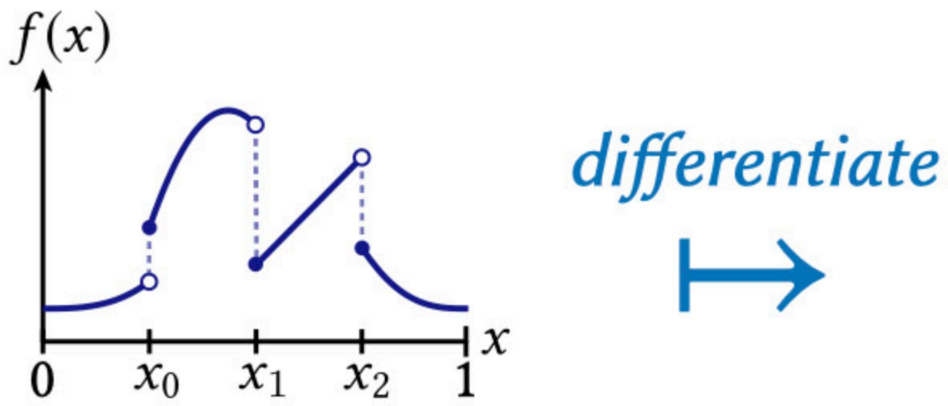

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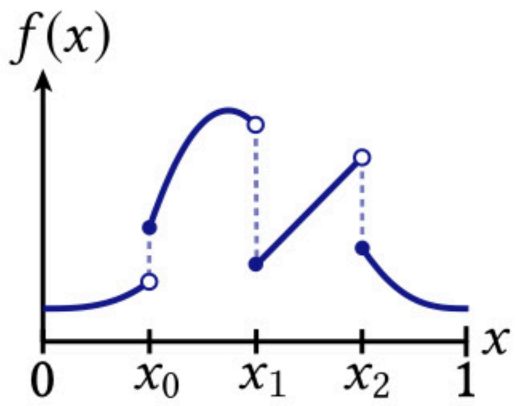
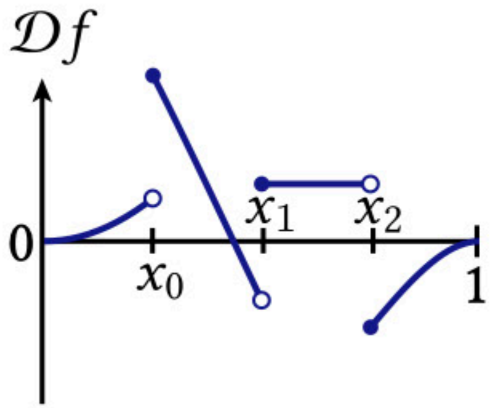


differentiate $\mapsto$

continuous part



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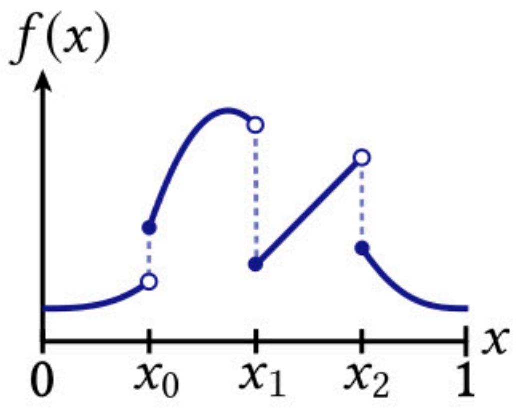

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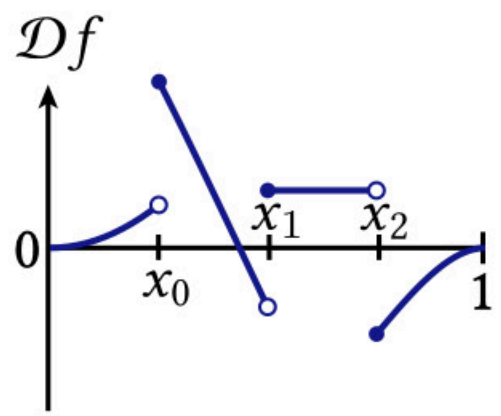

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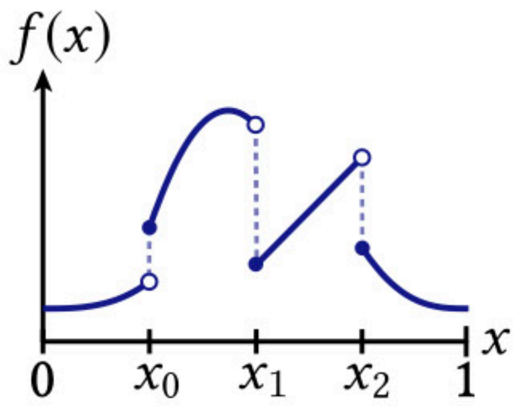

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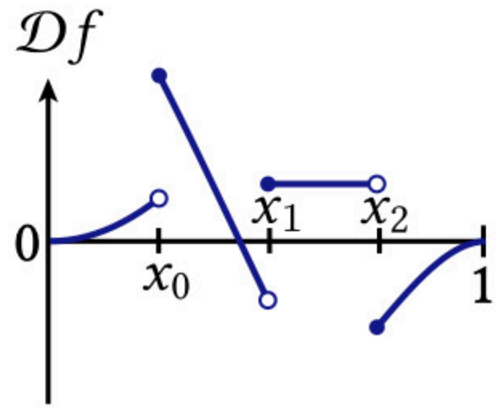
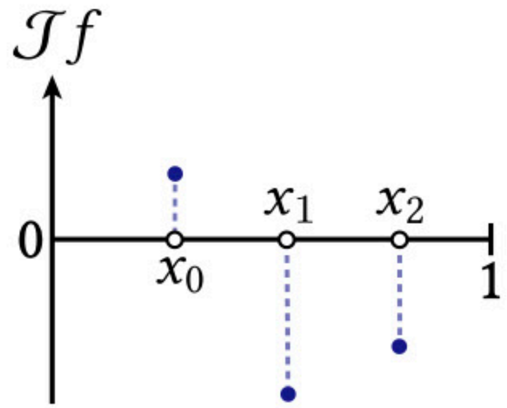


+

"jump part"



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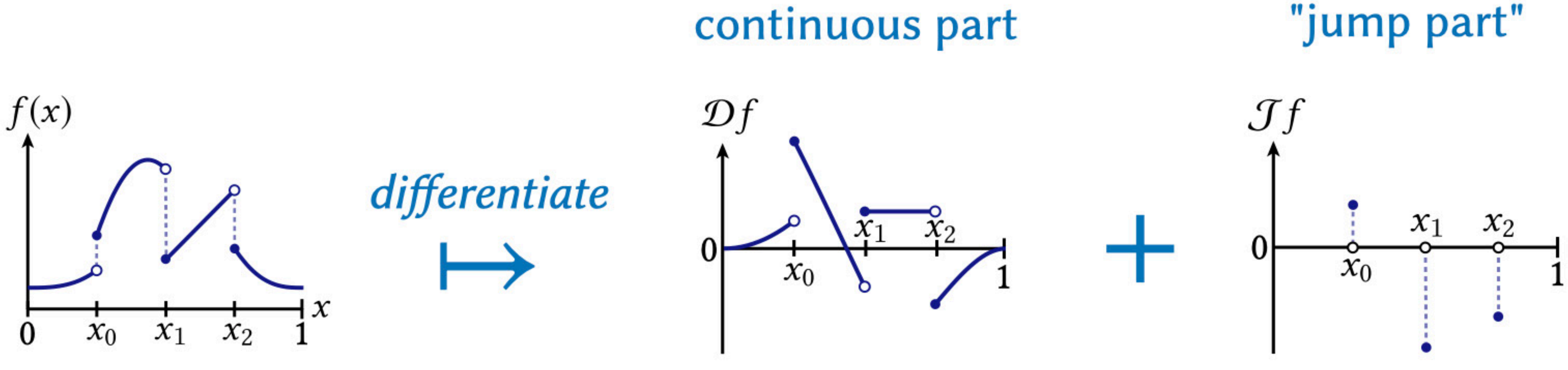

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$\mathcal{J}f = \Sigma_i \Lambda_i \delta_{x_i}$

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For a **jump harmonic** function f, the *Darboux derivative* $\omega := \mathcal{D}f$ "forgets" jumps:

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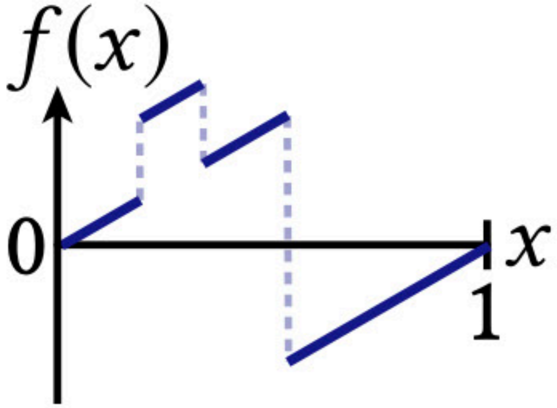

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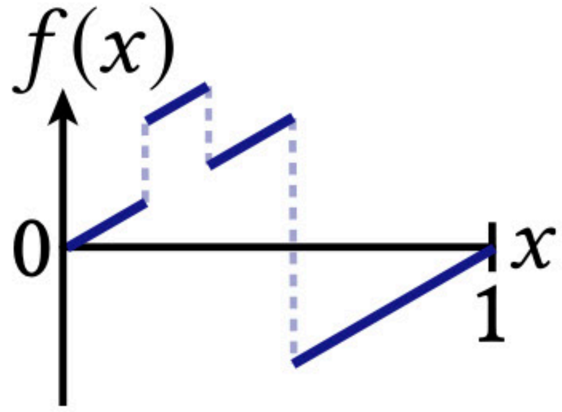

differentiate

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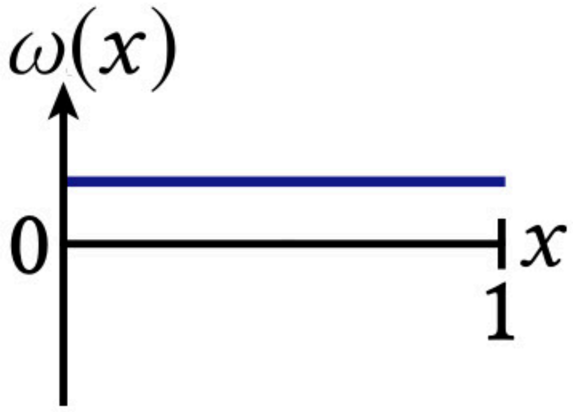
Darboux derivative



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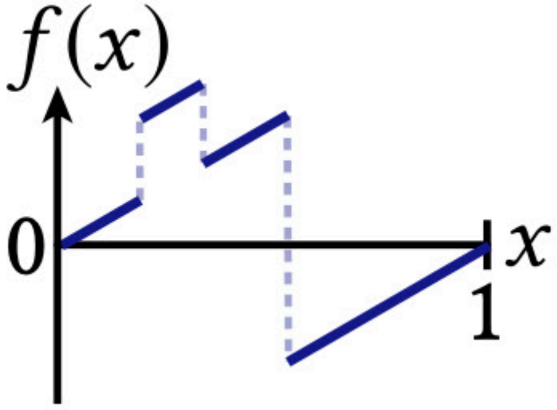

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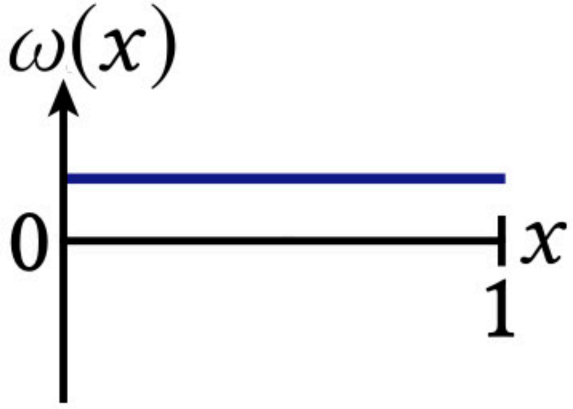

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$+$ jump derivative

Integrating the Darboux derivative

We can only integrate "up to jumps":

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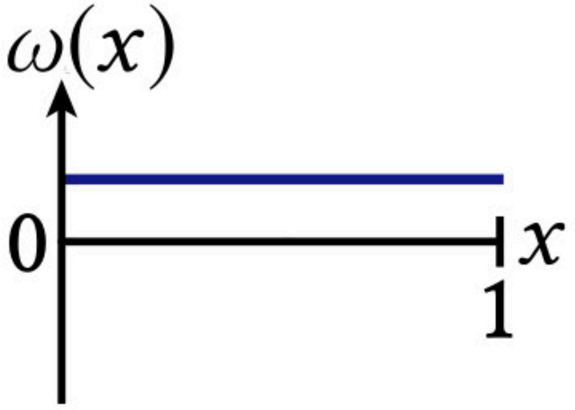

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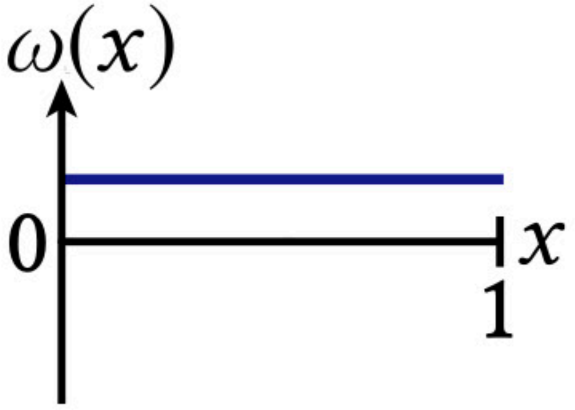

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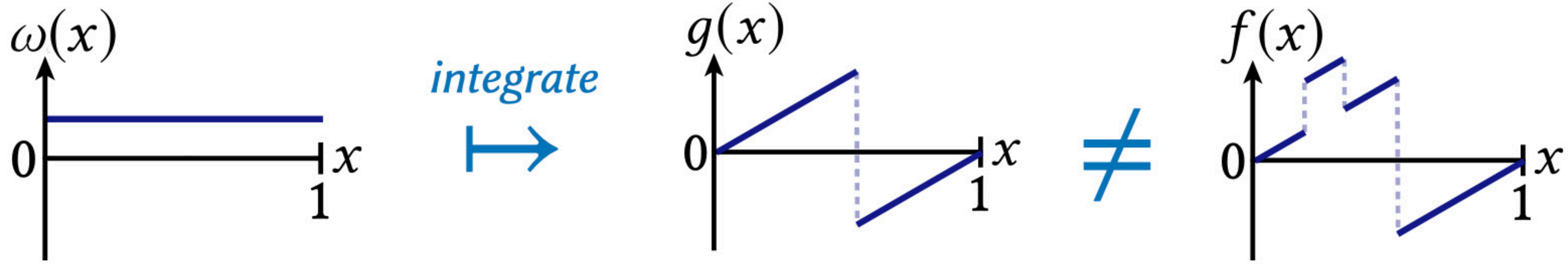

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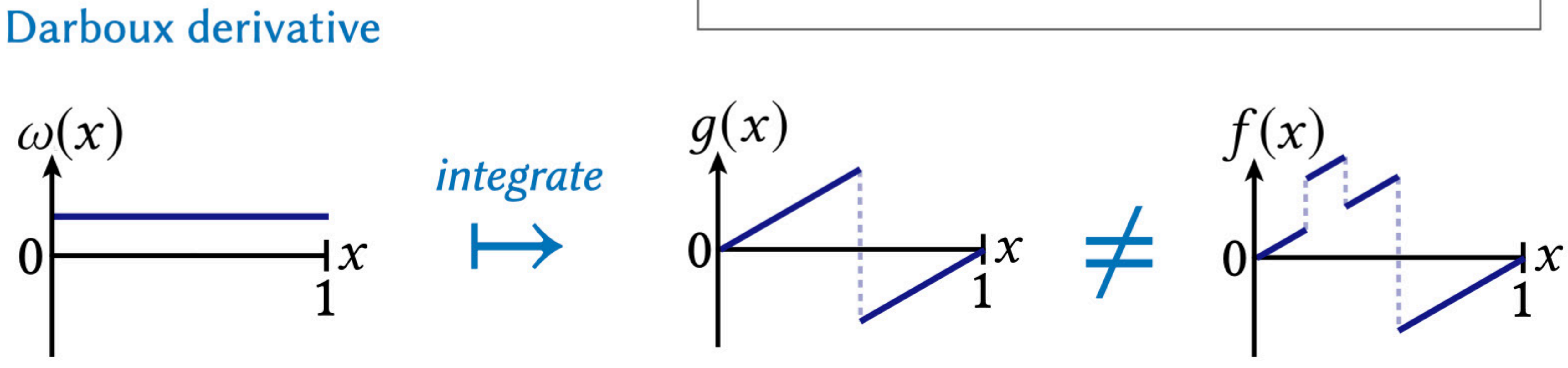
Moral: To map from derivatives back to curves, we can integrate ω — and choose the jumps!



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2D jump harmonic functions — same story

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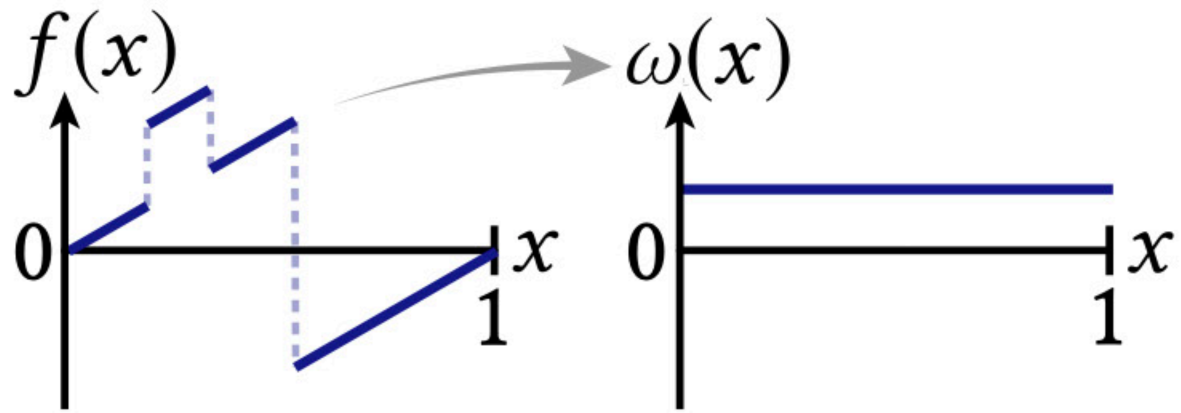

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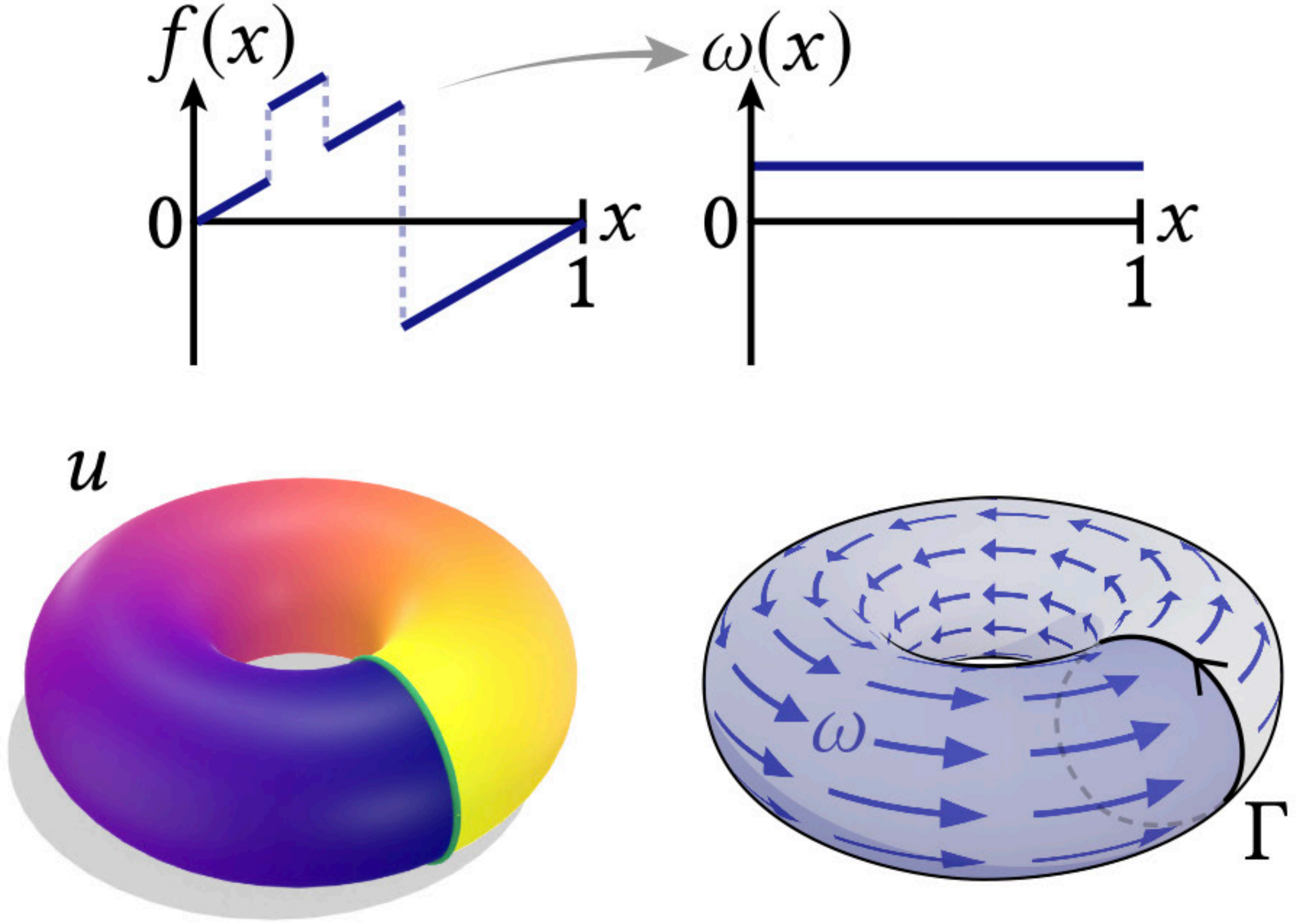

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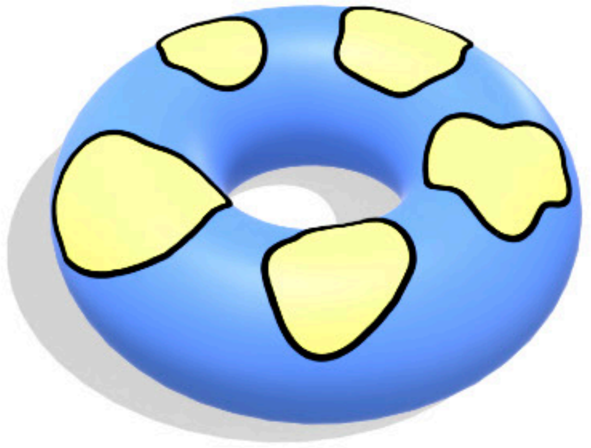
If $\omega = 0$, then u is piecewise constant $\Rightarrow$ u is already a valid region labeling.



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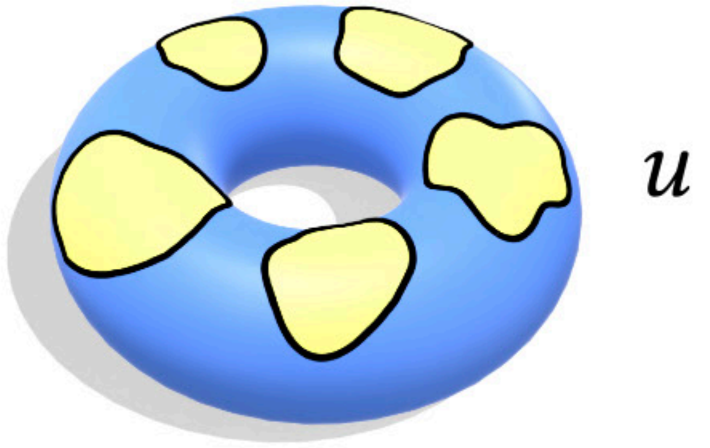
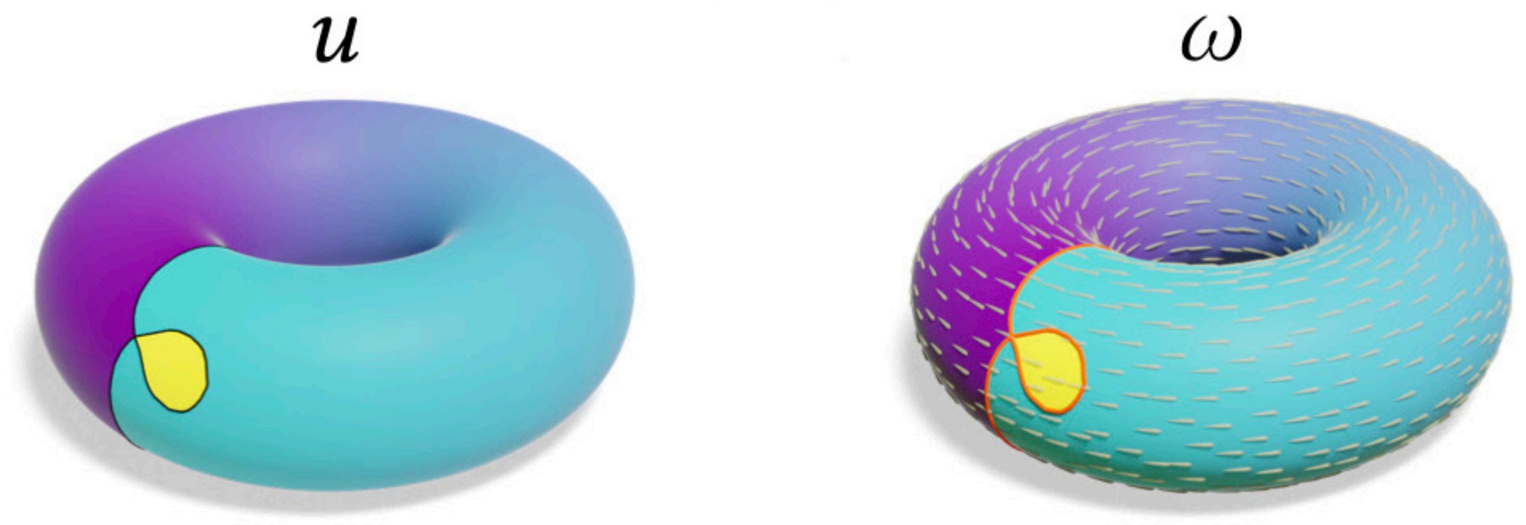


Otherwise, Γ has nonbounding components:



Derivative decomposition

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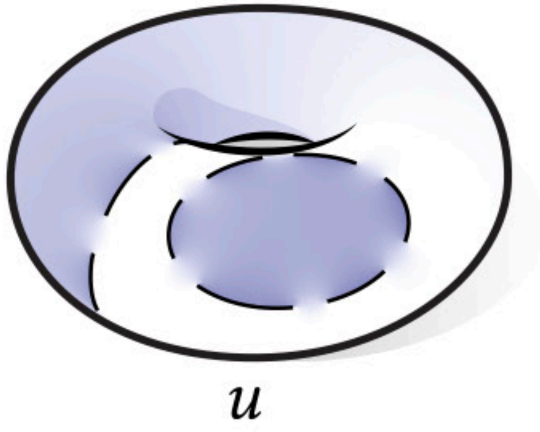
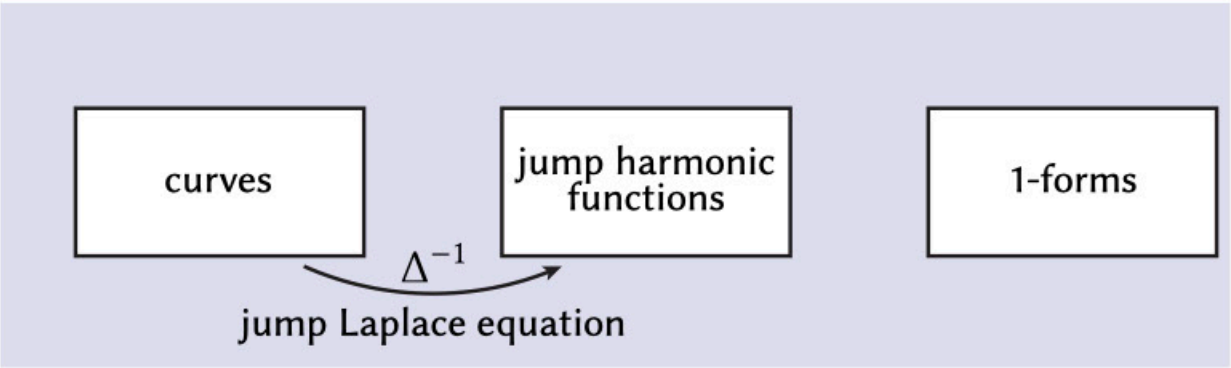
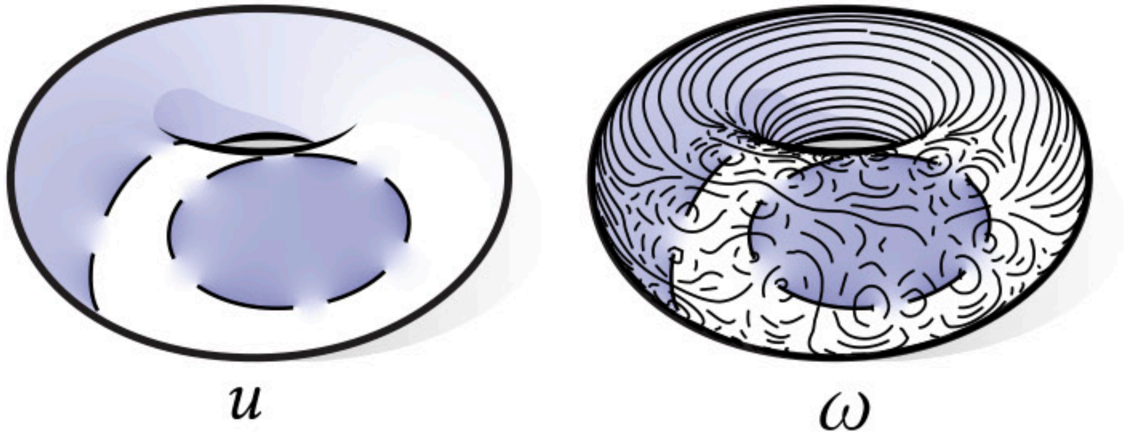
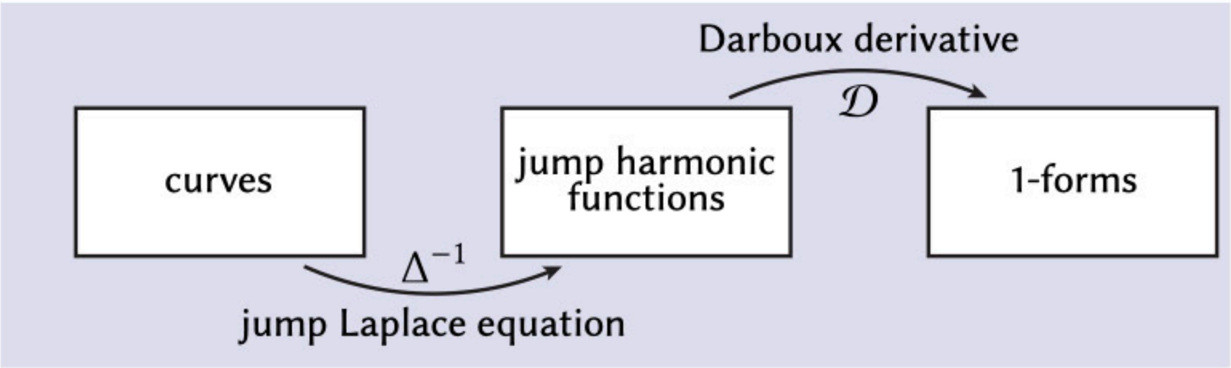




$$\Gamma \xrightarrow{\Delta^{-1}} u$$

Derivative decomposition





$$\Gamma \xrightarrow{\Delta^{-1}} u \xrightarrow{\mathcal{D}} \omega$$

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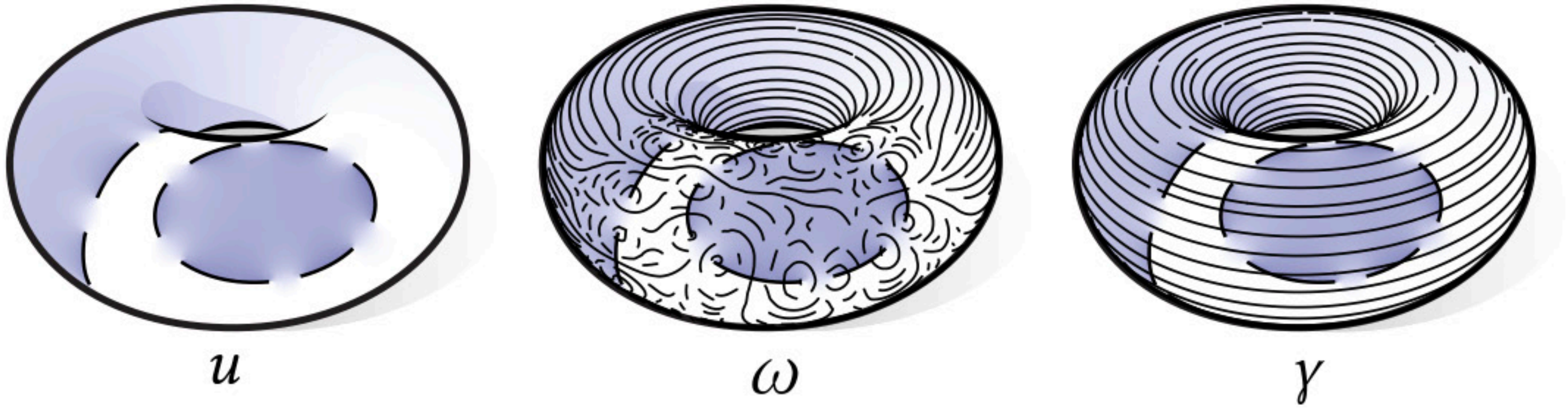
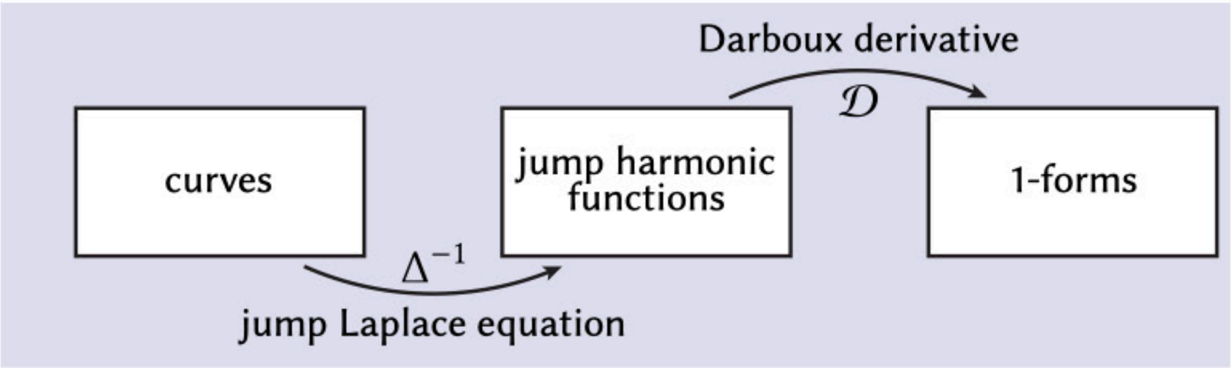


Hodge decomposition:

$$\omega = d\alpha + \delta\beta + \gamma$$

γ is a **harmonic 1-form**



$$\Gamma \xrightarrow{\Delta^{-1}} u \xrightarrow{\mathcal{D}} \omega \longrightarrow \gamma$$

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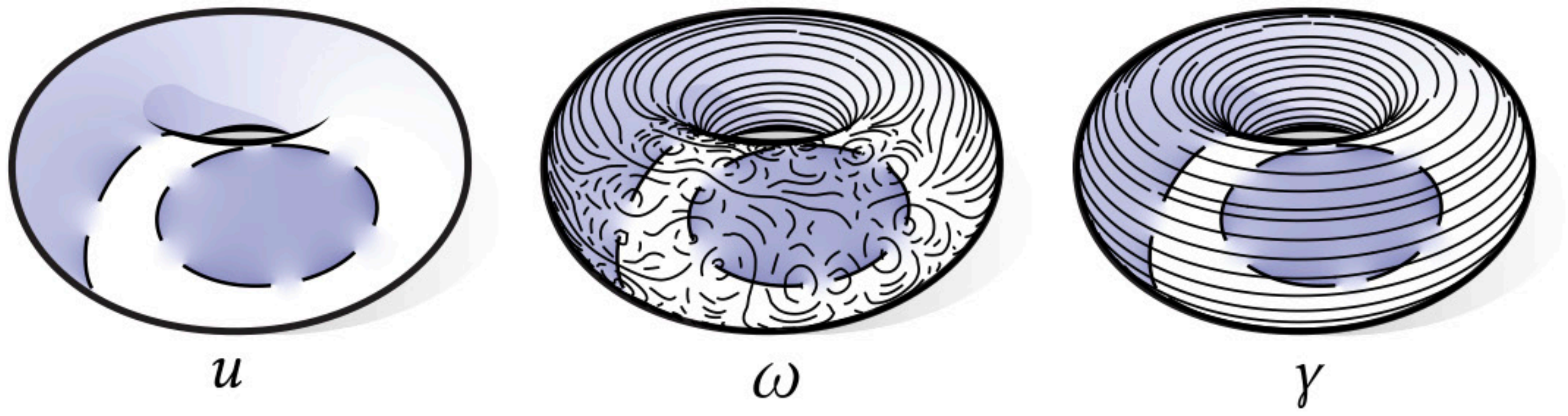

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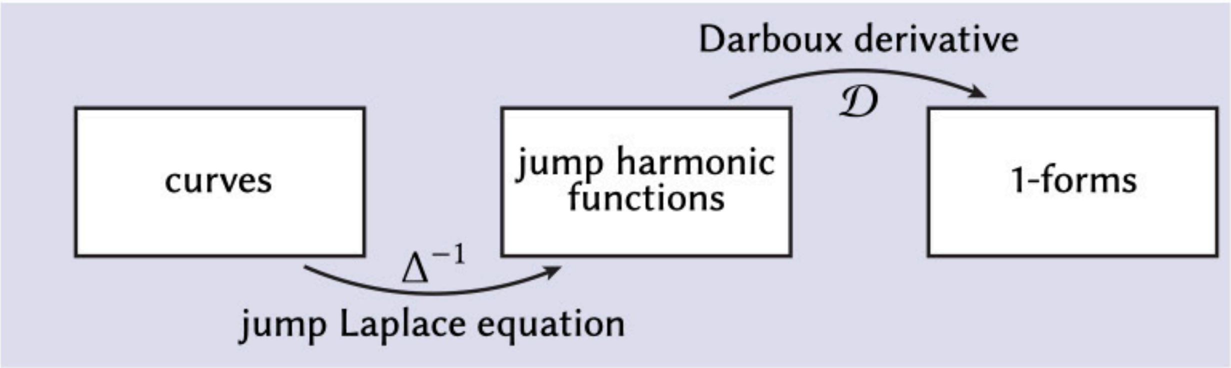
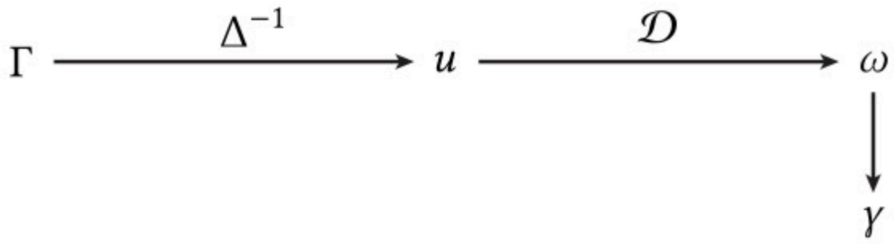
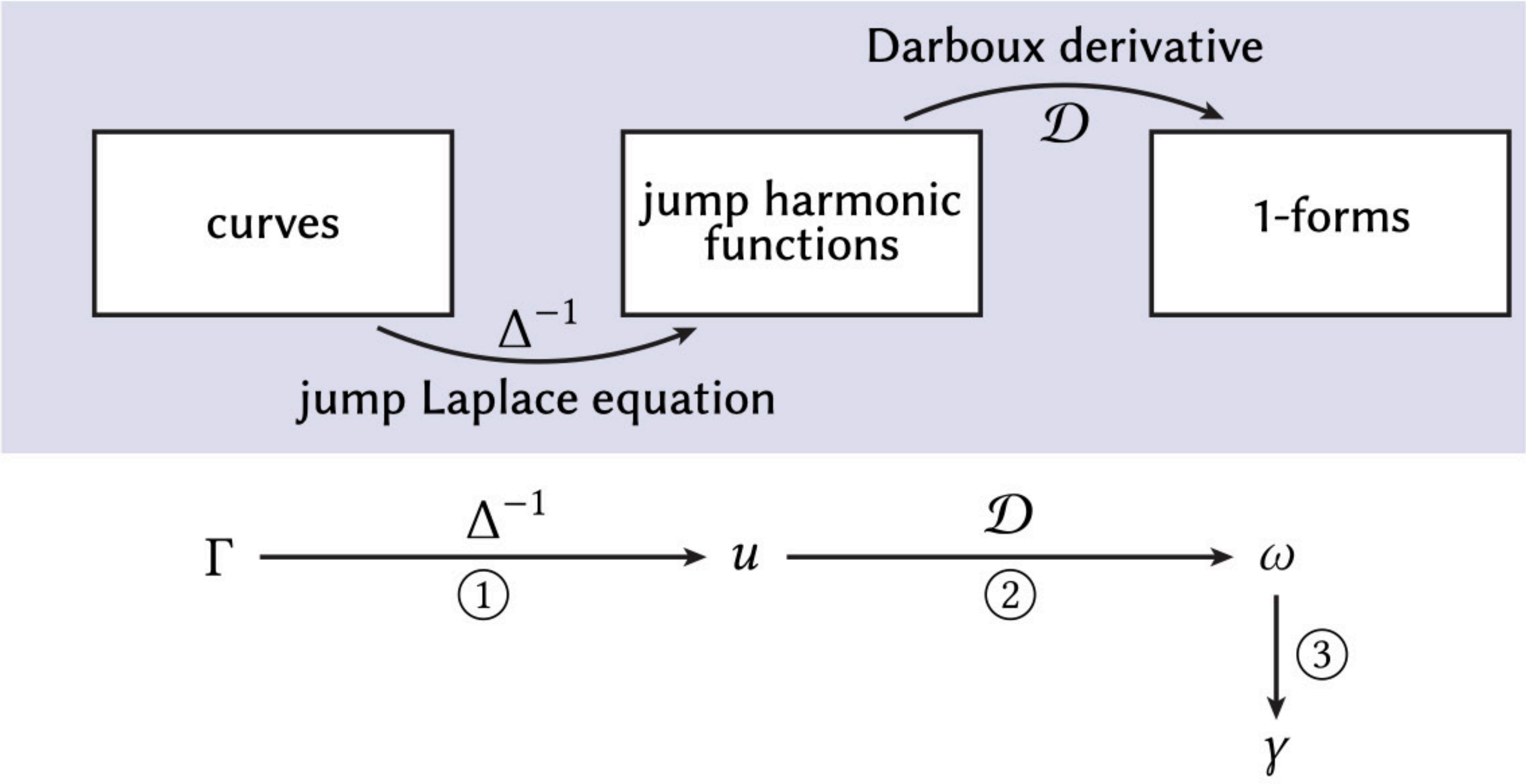




Nonbounding components of Γ are encoded by the harmonic component γ of ω.

More formally: (Non)bounding components of Γ correspond to 1-forms (non)congruent to zero in the *first cohomology group* $H^1(M) = \ker(d_1) \setminus \operatorname{im}(d_0)$.

Derivative decomposition → curve decomposition



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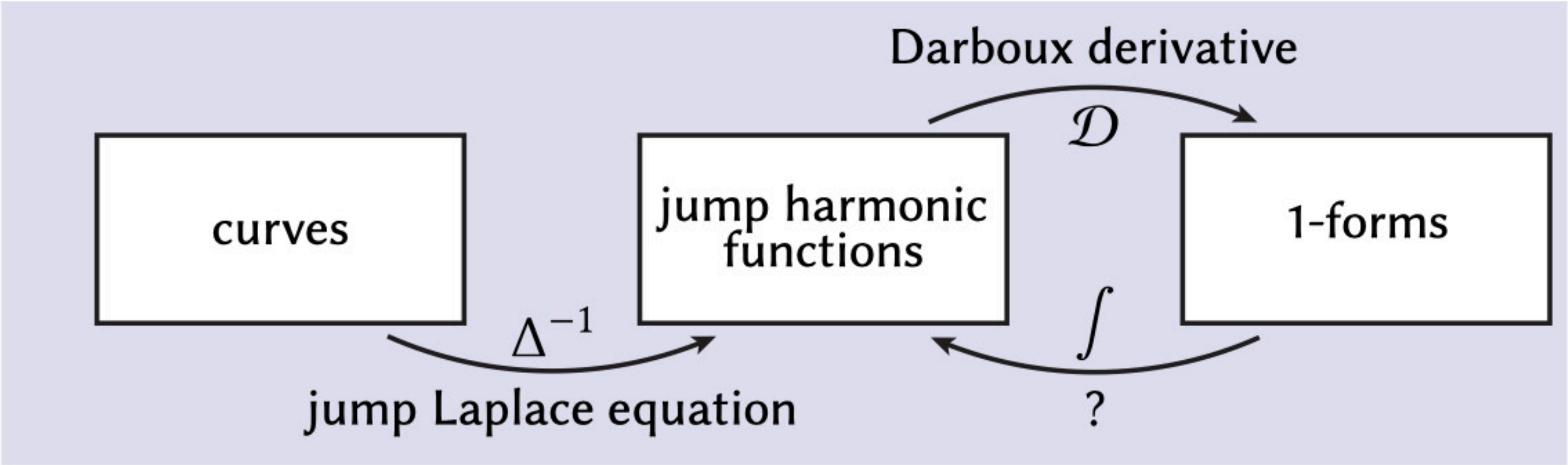

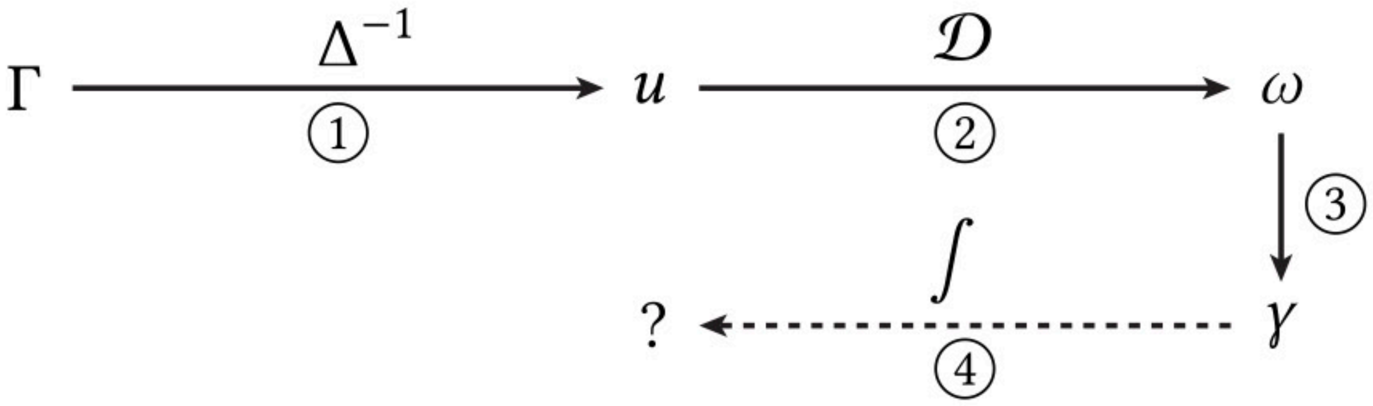

1-forms → jump harmonic functions

Search for a scalar potential v that could have generated γ.

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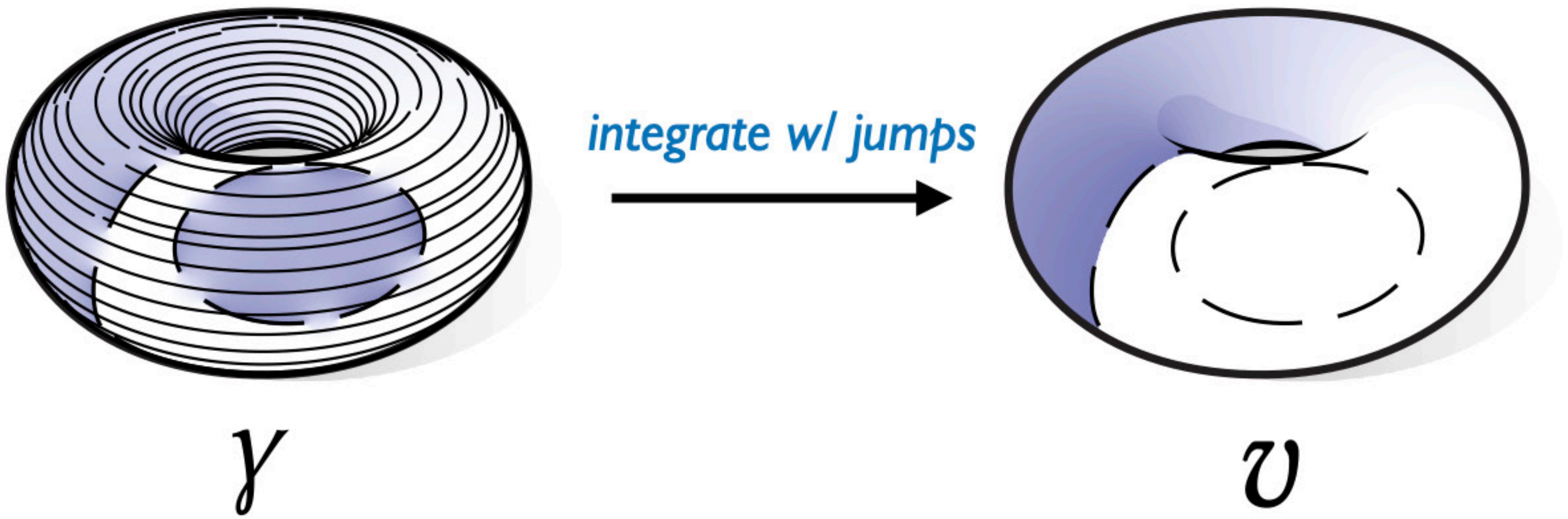

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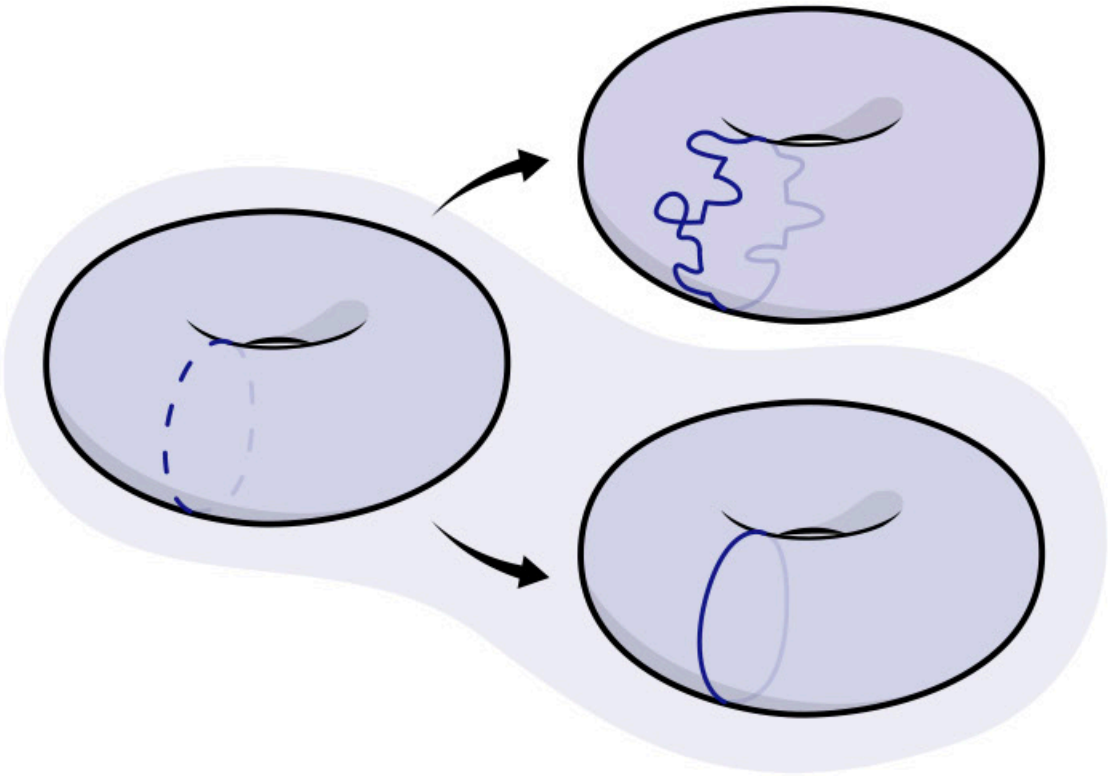

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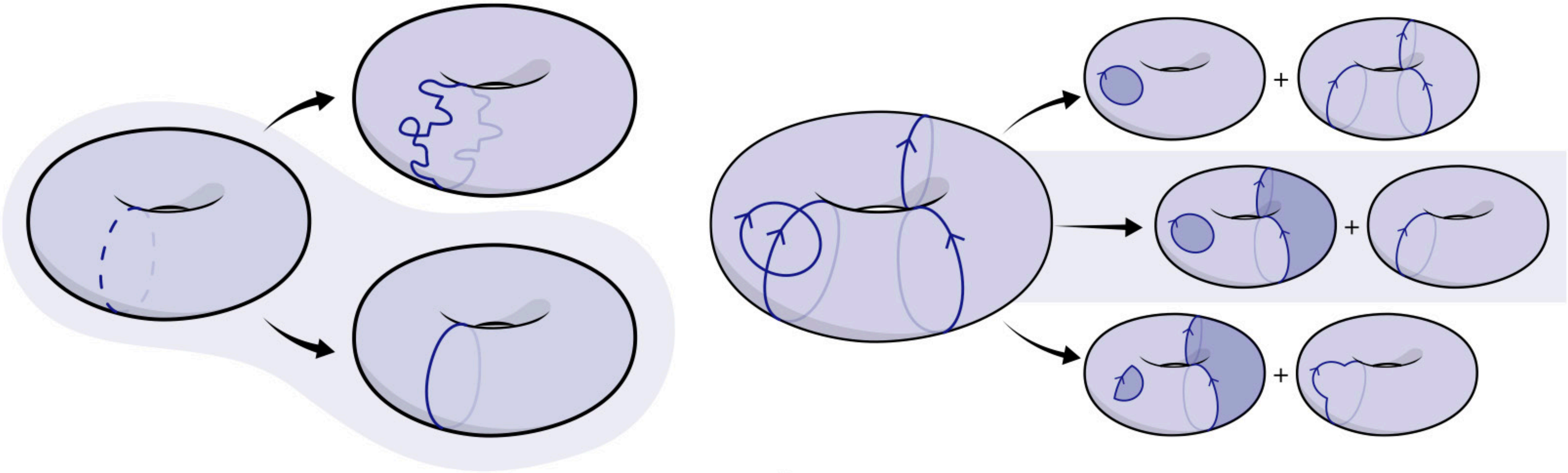
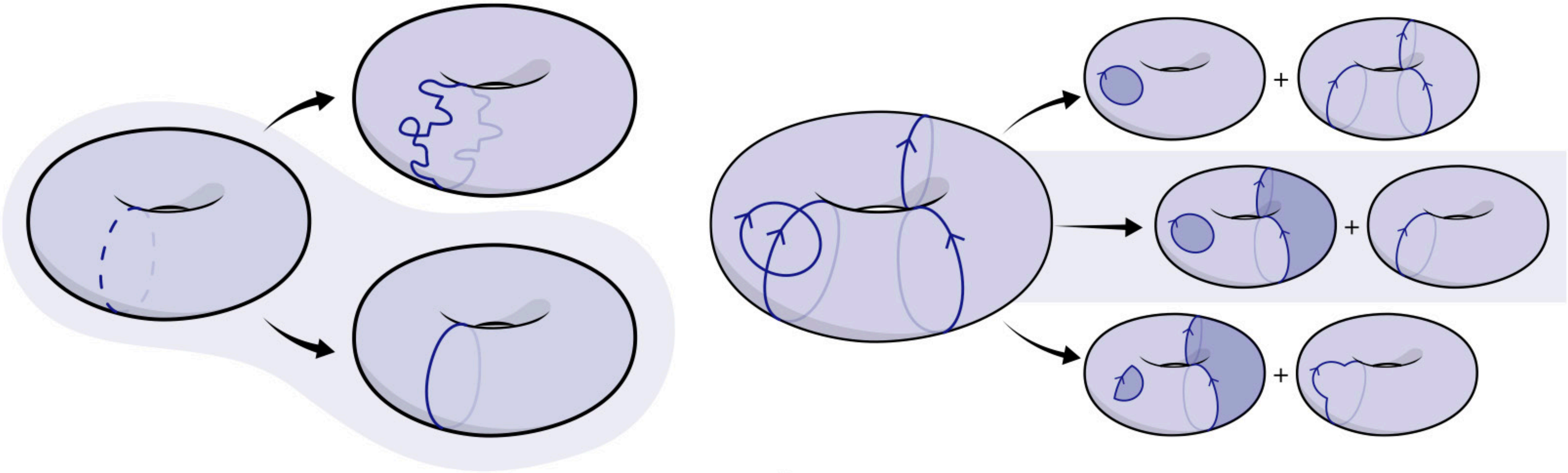


1-forms → jump harmonic functions

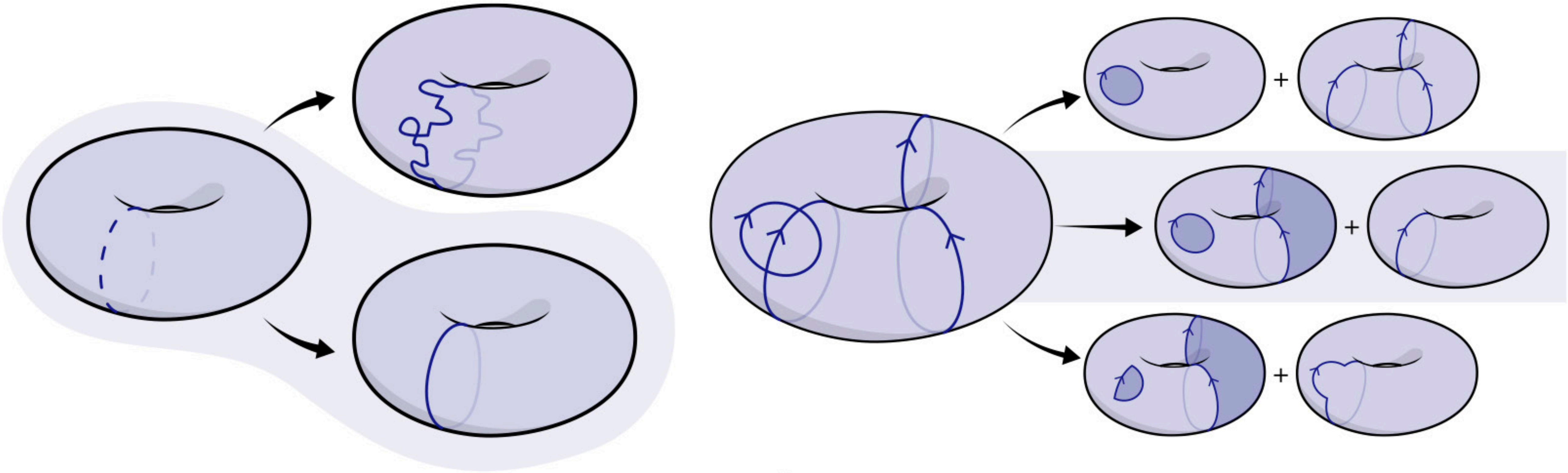
$$\min_{v\colon M\to\mathbb{R}} \int |\text{the jumps not across } \Gamma| + \varepsilon \int |\text{the jumps across } \Gamma|$$



1-forms → jump harmonic functions

penalize jumps

$$\min_{v:\, M \to \mathbb{R}} \boxed{\int |\text{the jumps not across } \Gamma|} + \varepsilon \int |\text{the jumps across } \Gamma|$$



1-forms → jump harmonic functions

penalize jumps | *smaller penalty across Γ*

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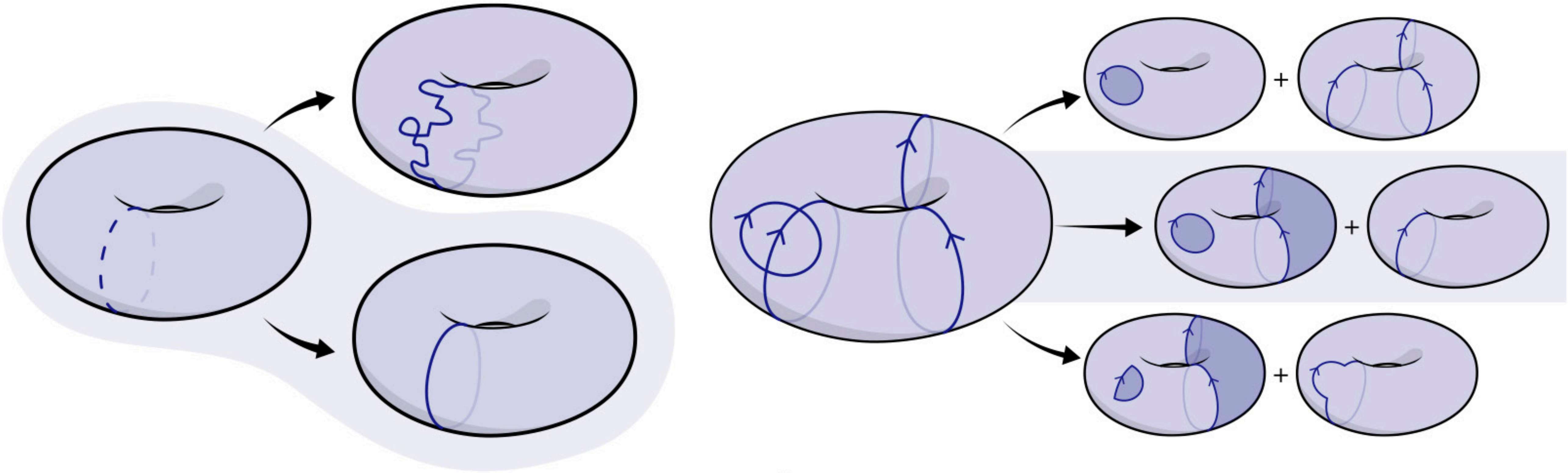

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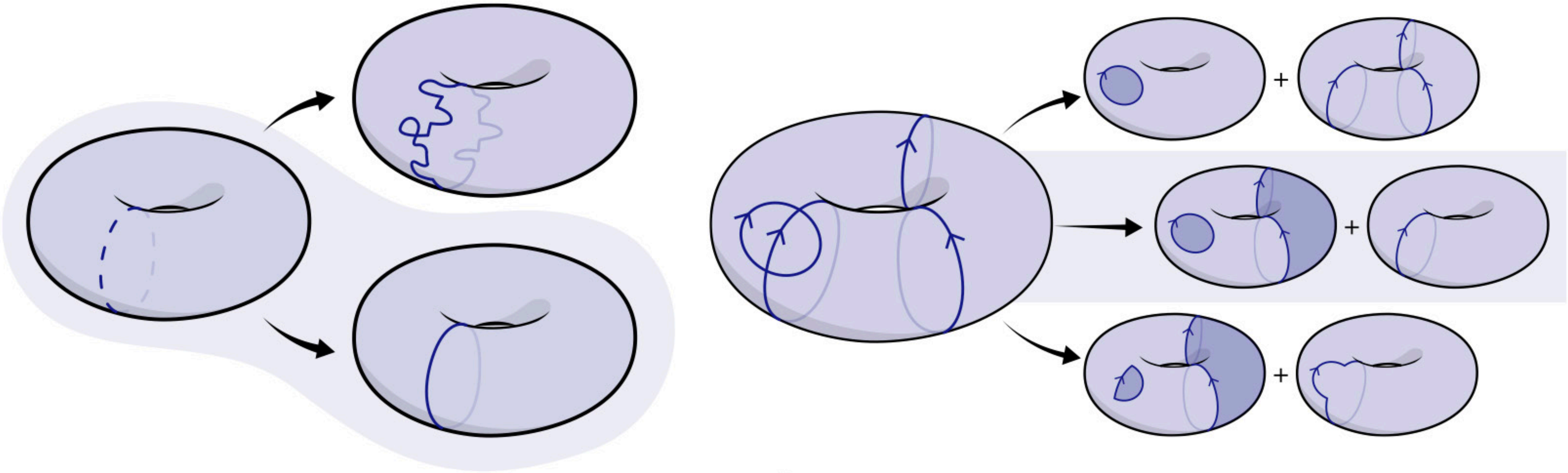

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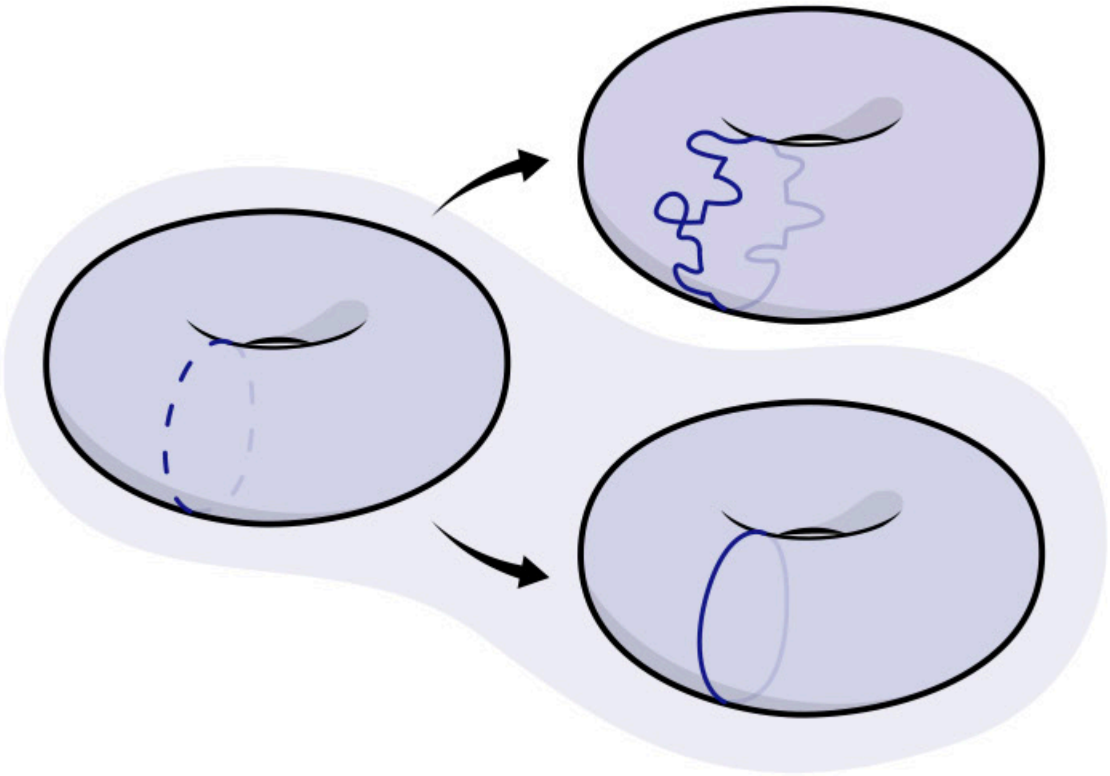

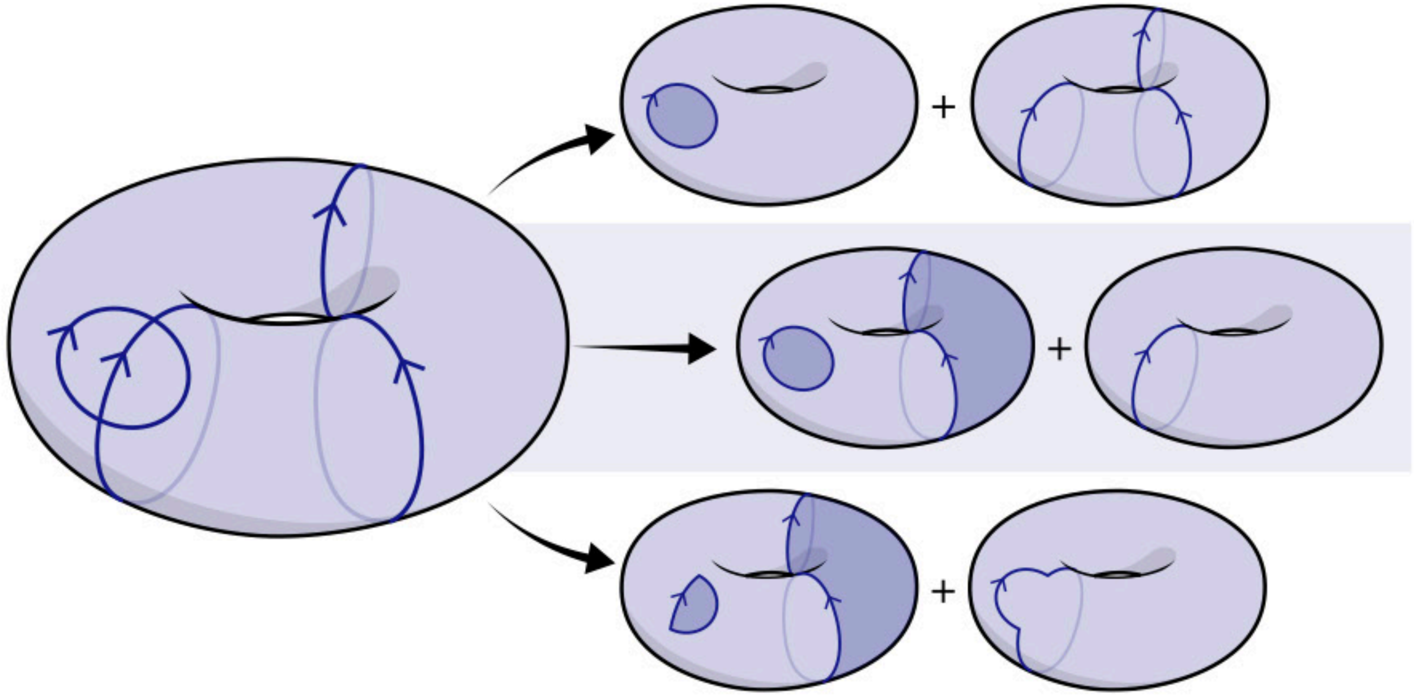

1-forms → jump harmonic functions

penalize jumps — *smaller penalty across Γ*

pick shortest completion in homology class

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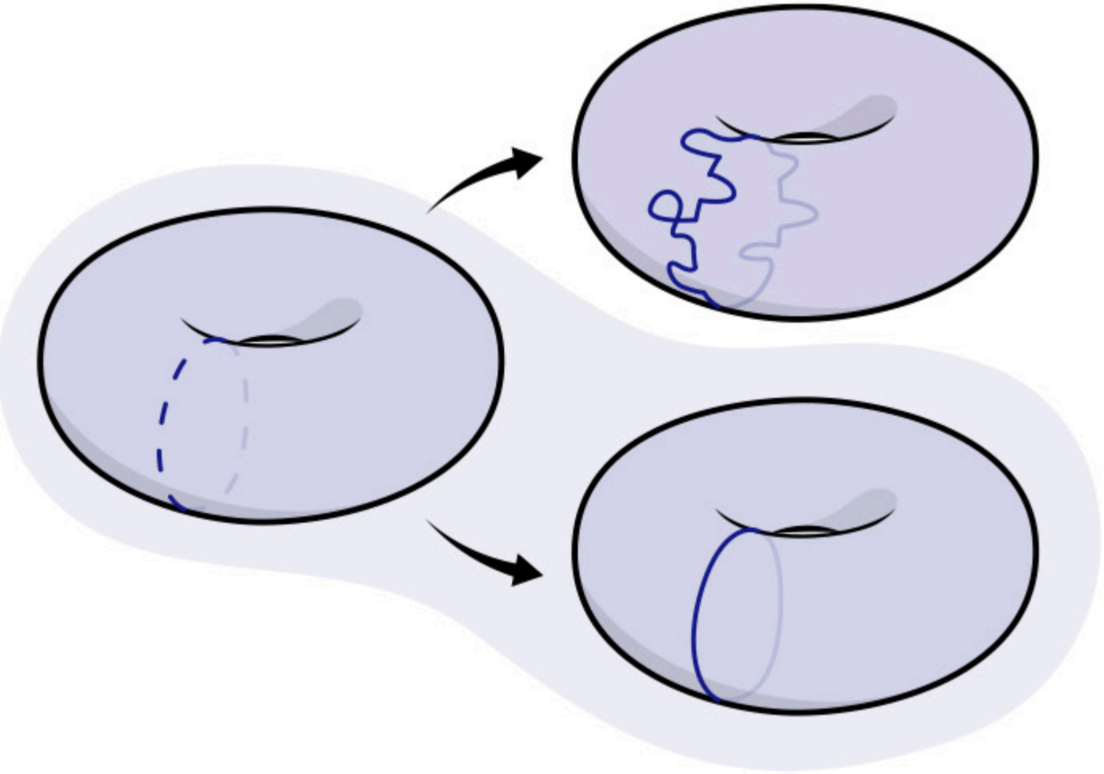

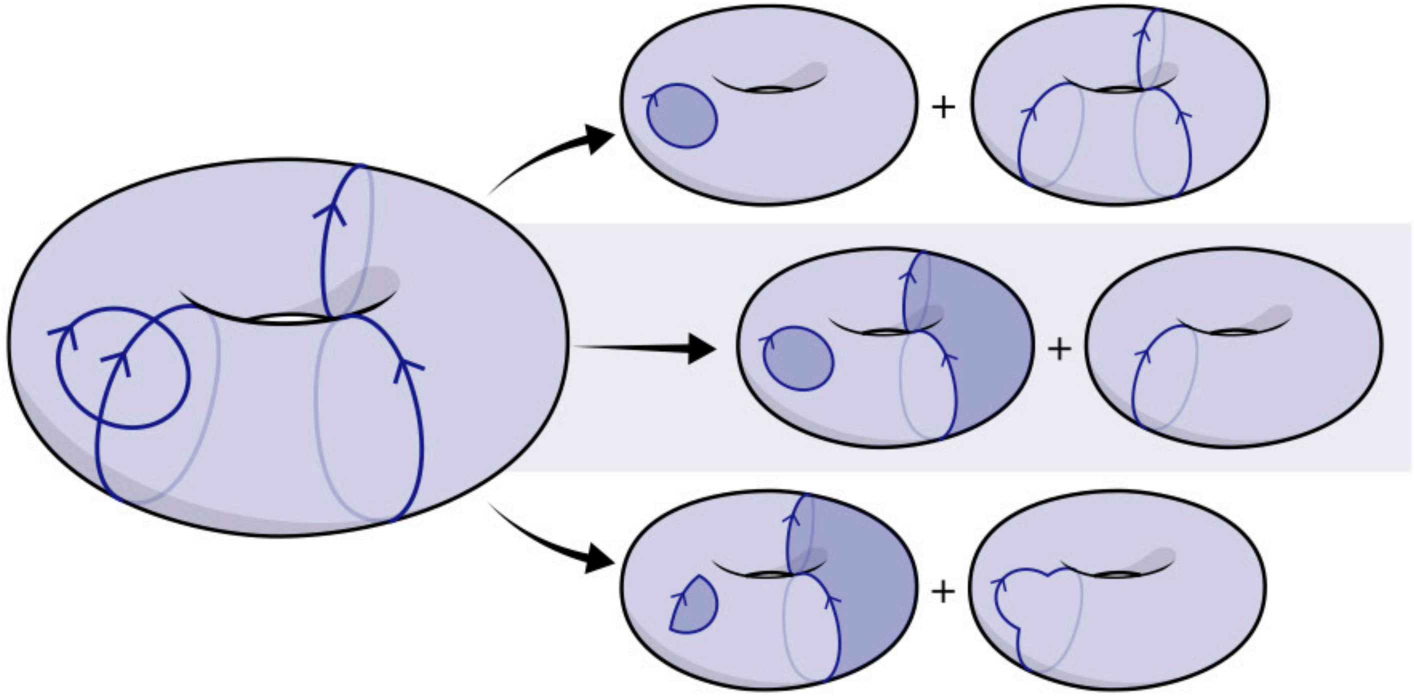

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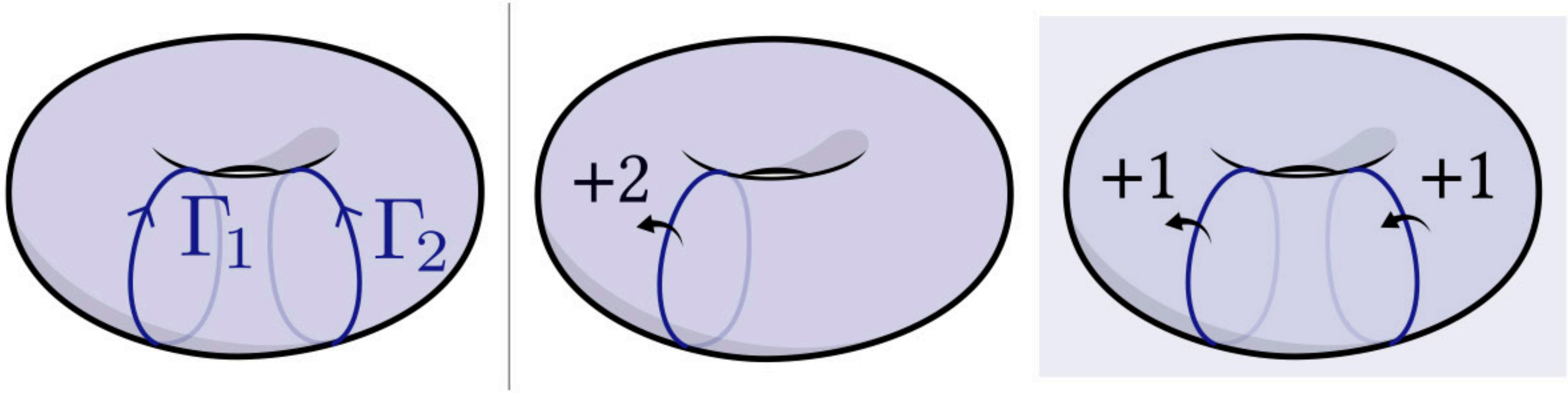

input v

1-forms → jump harmonic functions

pick shortest completion in homology class

penalize jumps — *smaller penalty across Γ*

$$\min_{v\colon M\to\mathbb{R}} \int |\text{the jumps not across } \Gamma| + \varepsilon \int |\text{the jumps across } \Gamma|$$

concentrate jumps across Γ

$$\text{subject to} \qquad \mathcal{D}v = \gamma$$

(co)homology constraint

$$0 \le \frac{v^+ - v^-}{u^+ - u^-} \le 1 \quad \text{on } \Gamma$$

no extra loops

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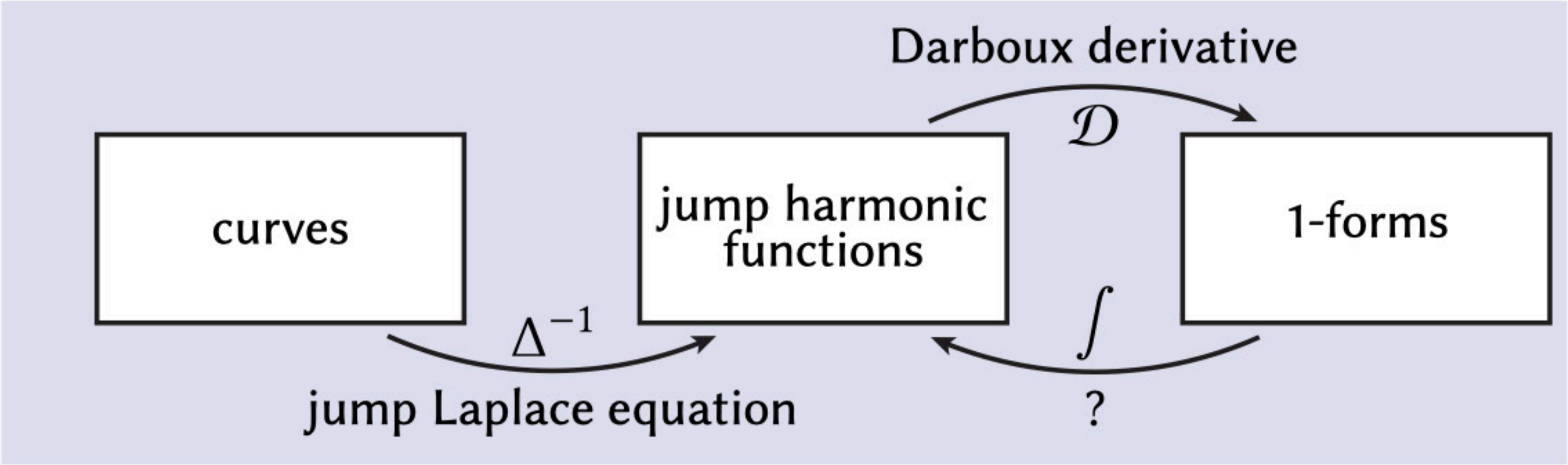
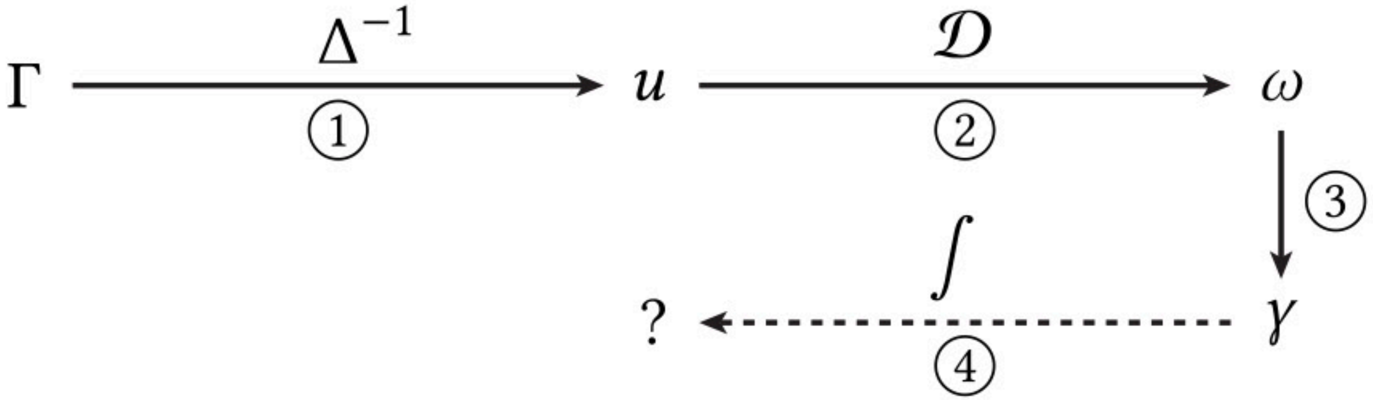
no extra loops

"residual function"

1-forms → jump harmonic functions





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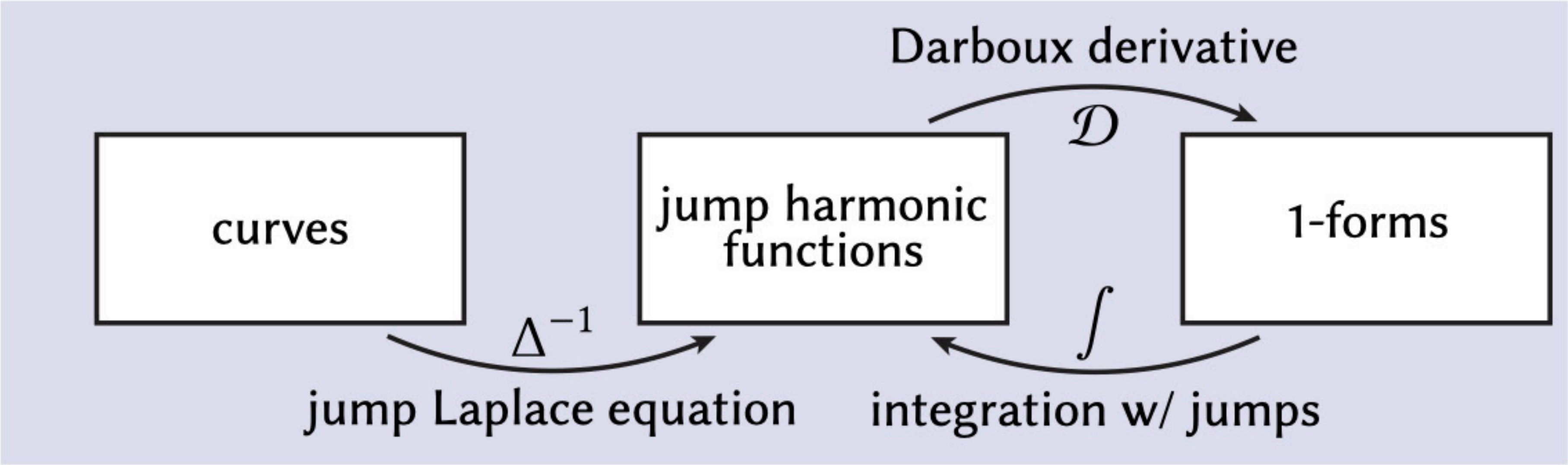

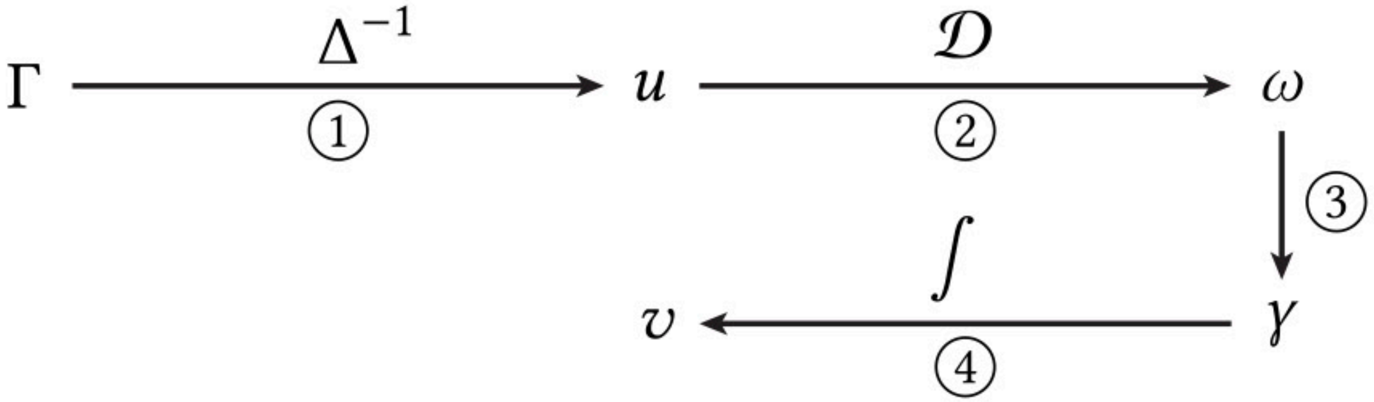

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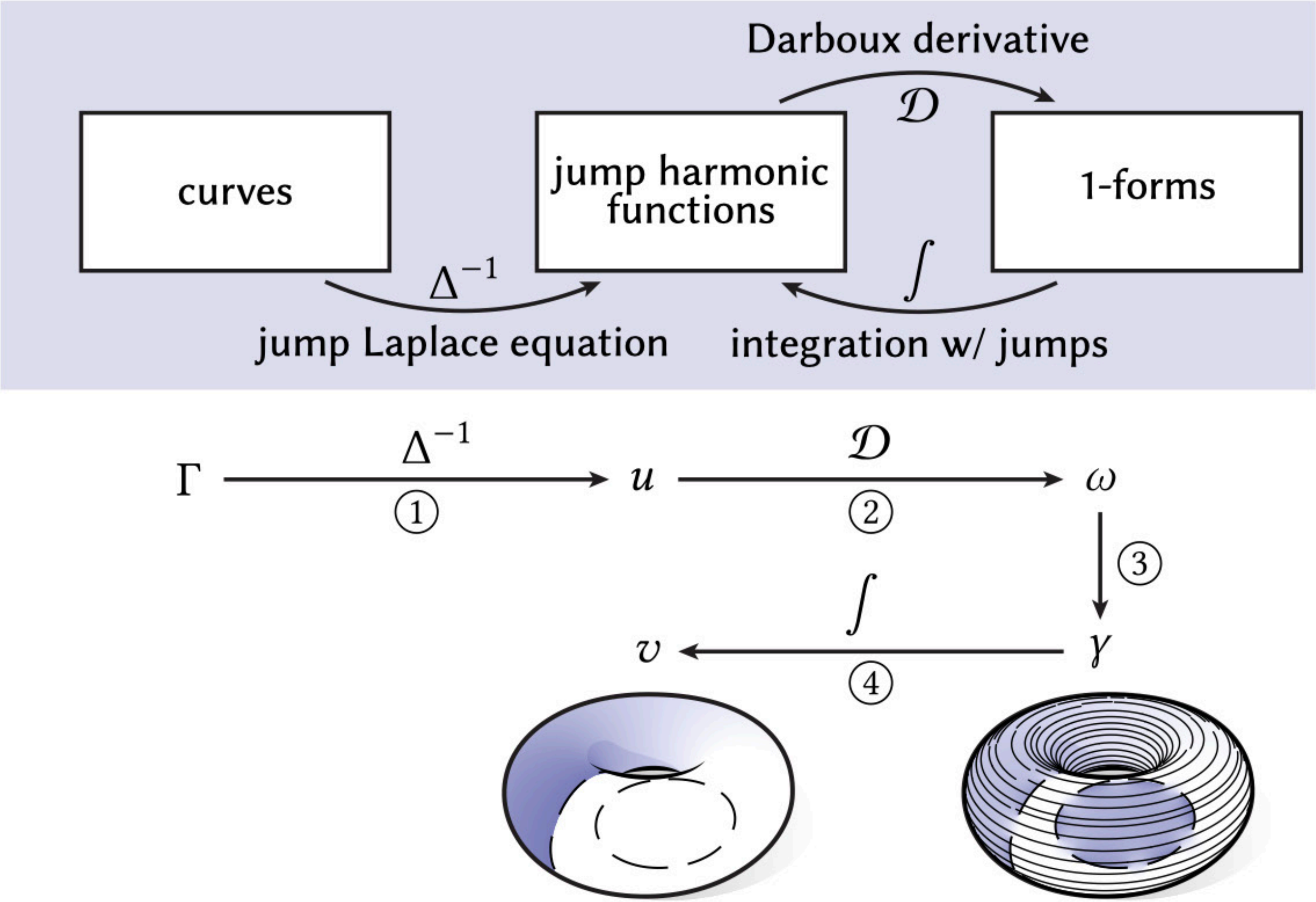
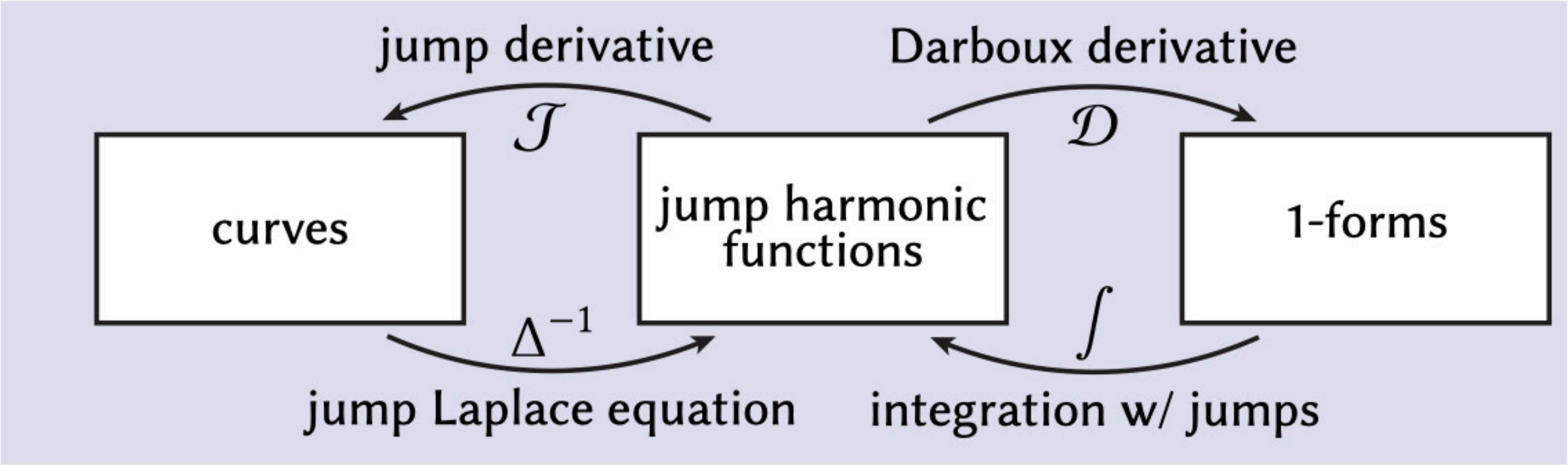
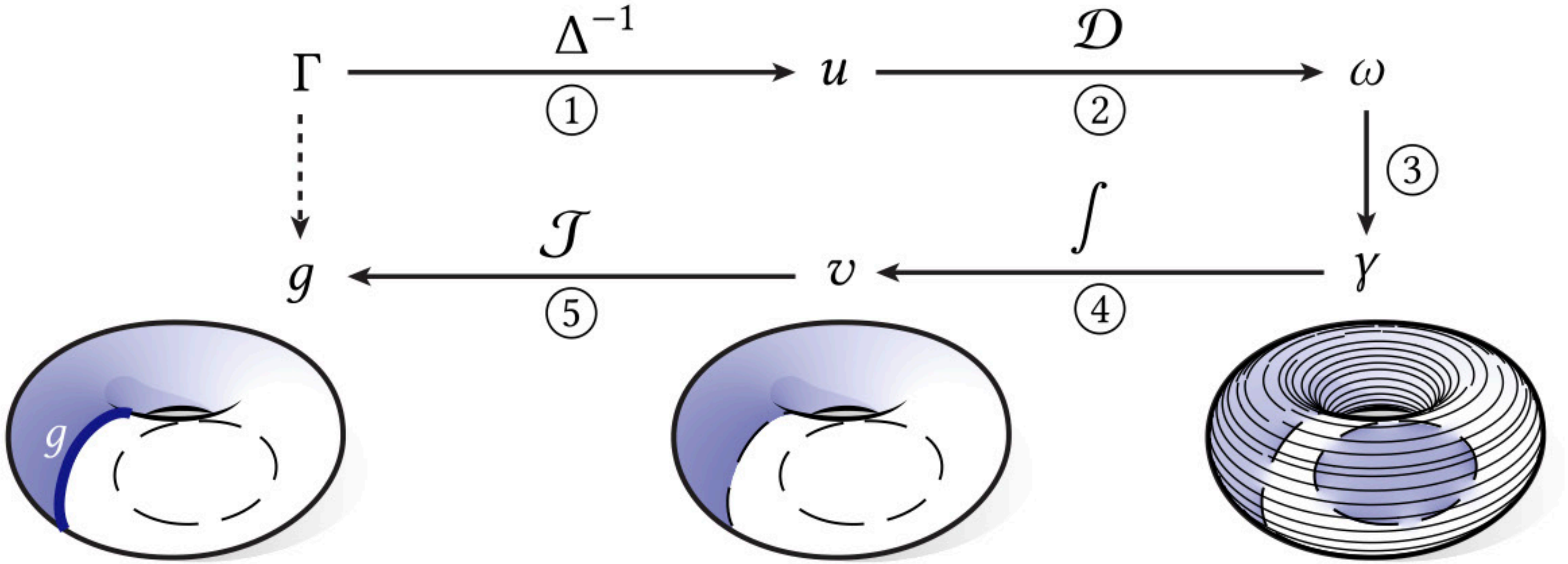


Jump harmonic function → curve decomposition





Winding number function

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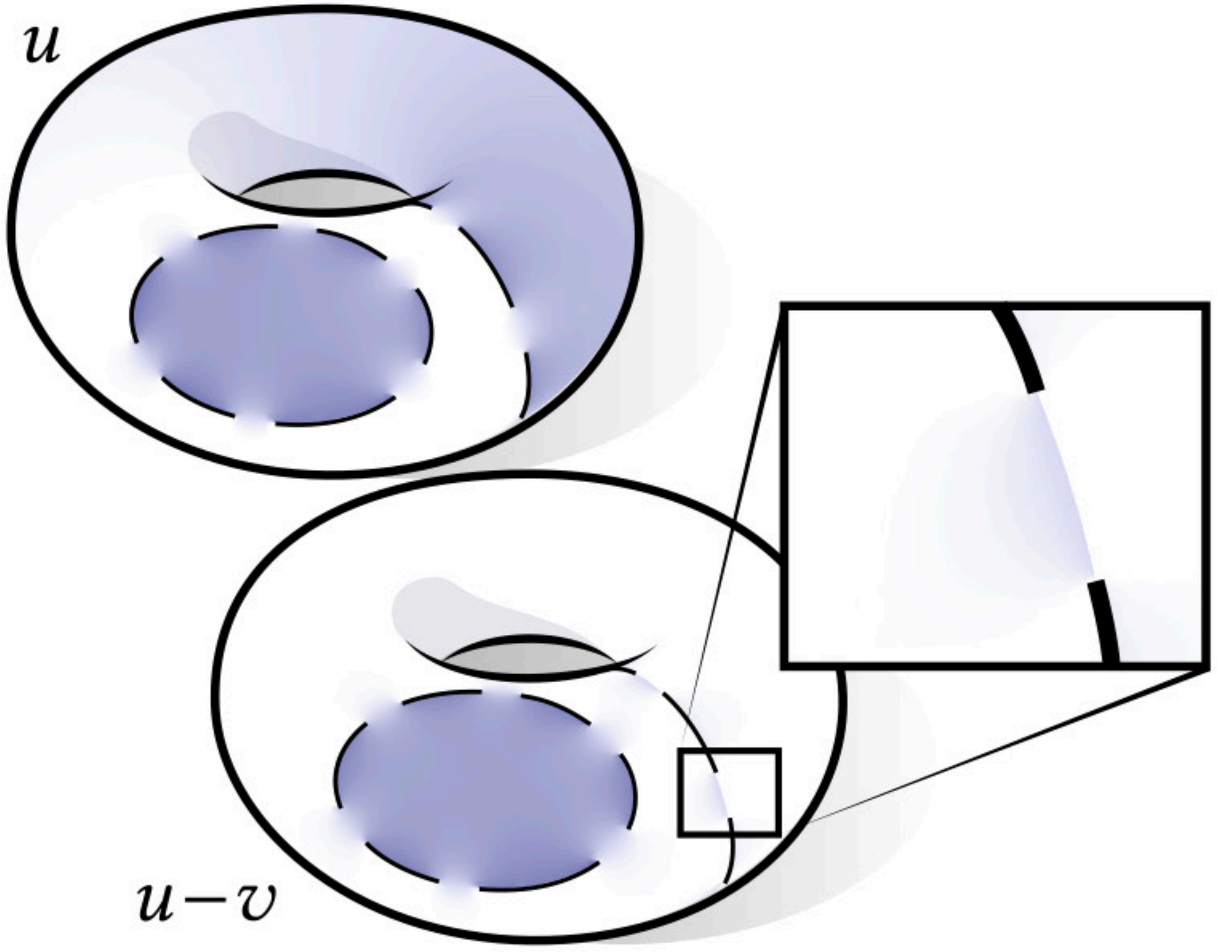
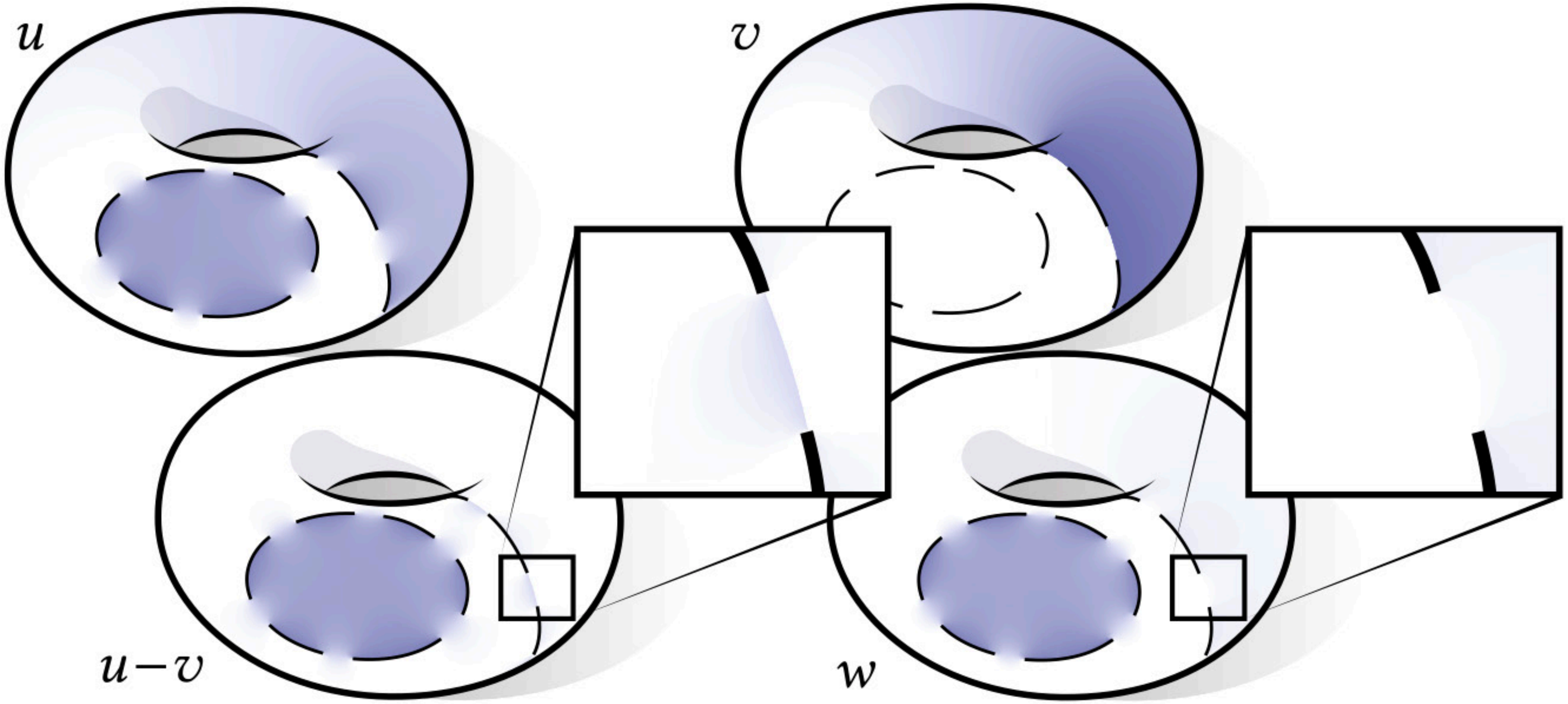
Simply subtracting the residual function
yields extraneous discontinuities.



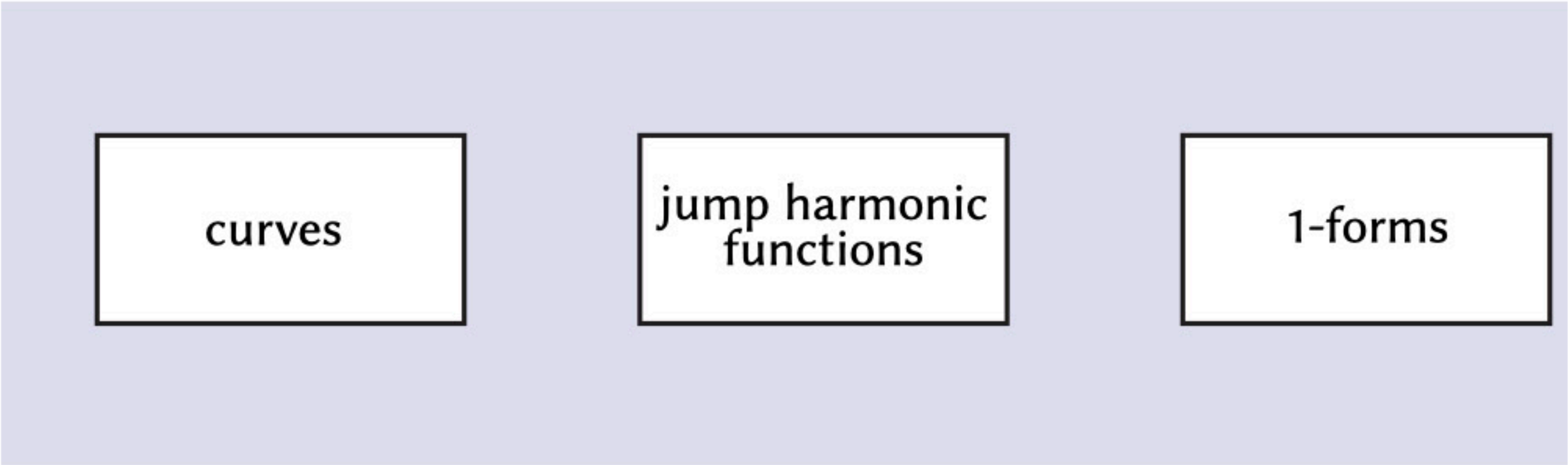
Winding number function

Simply subtracting the residual function yields extraneous discontinuities.

Solution: Solve for a new harmonic function w with jumps only across Γ.



Completing the round-trip



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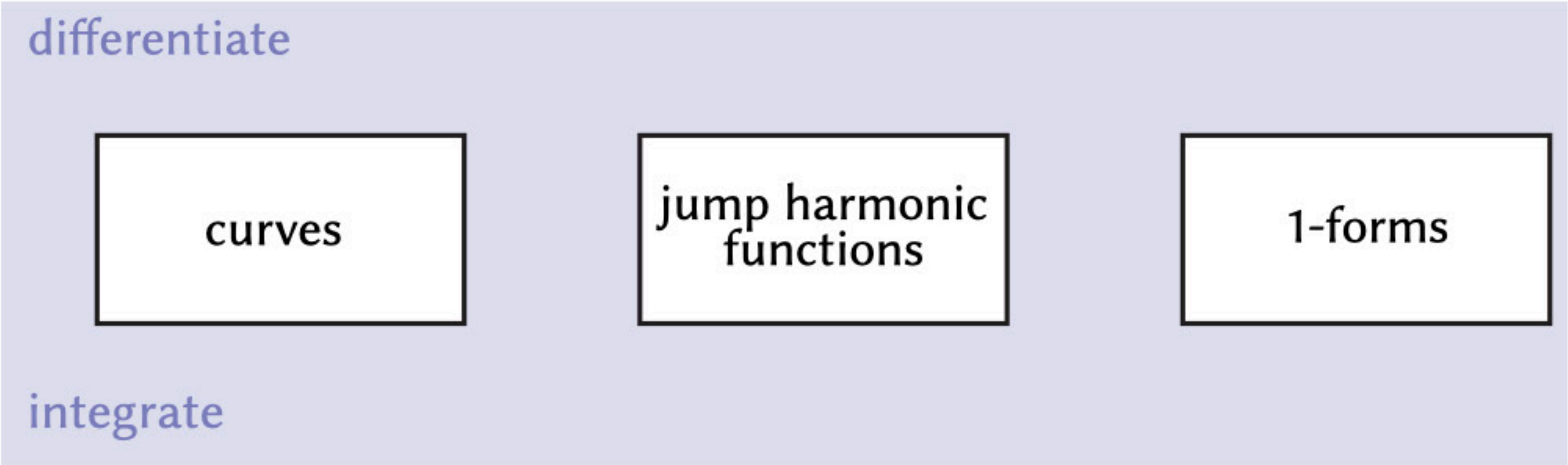


Completing the round-trip

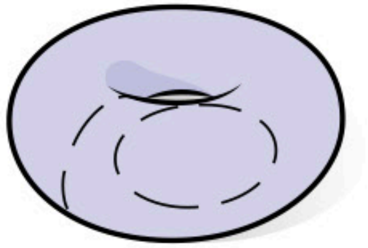
differentiate

curves

jump harmonic functions

1-forms

integrate

Input: 

Γ

Completing the round-trip

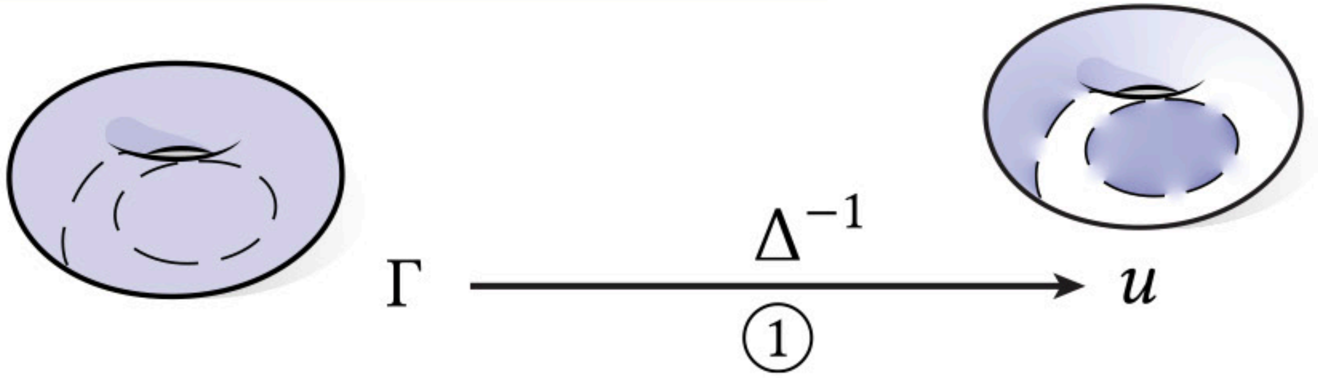
differentiate

curves → jump harmonic functions → 1-forms

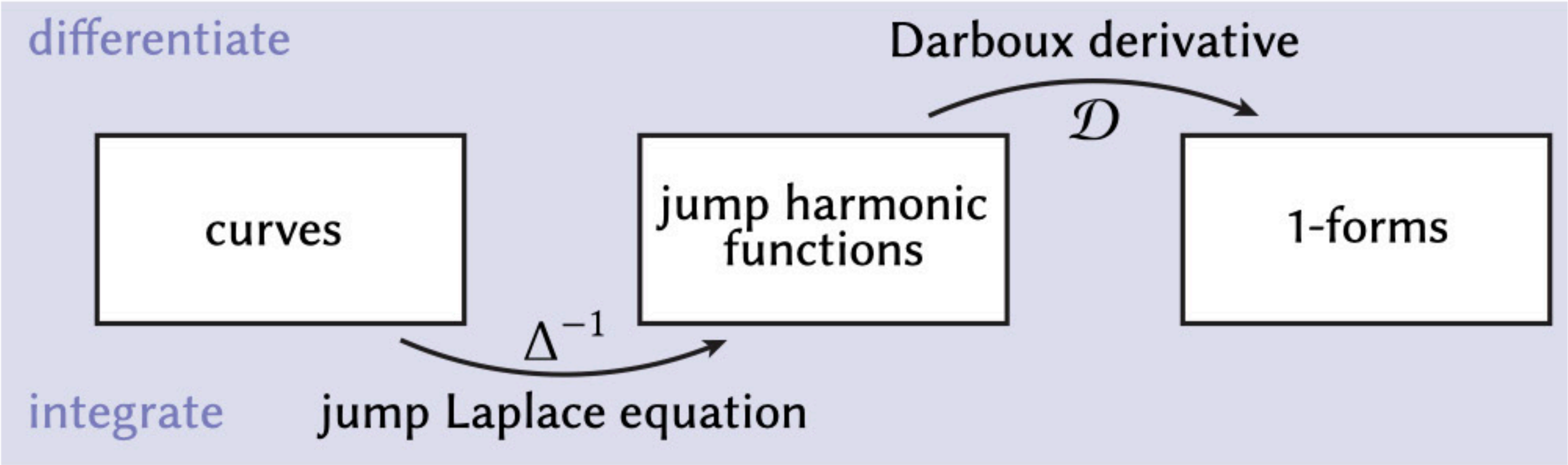
Δ^{-1}

integrate — jump Laplace equation

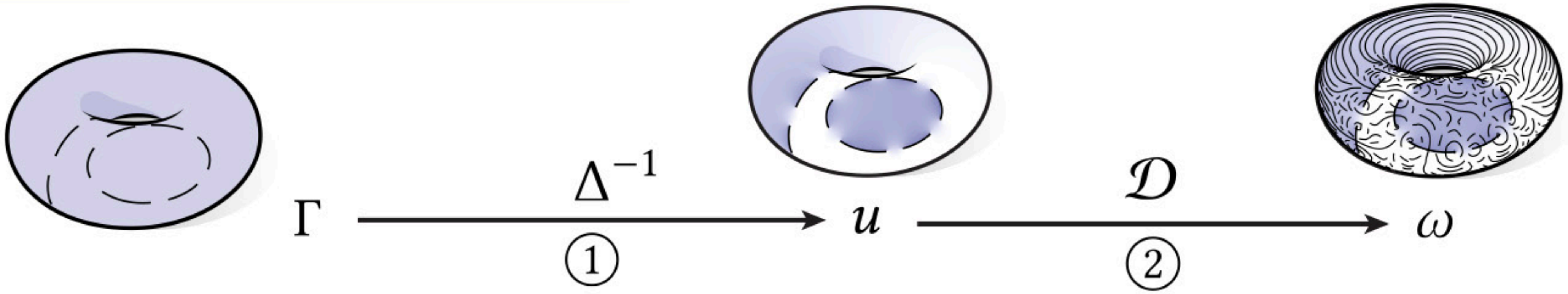
Input:



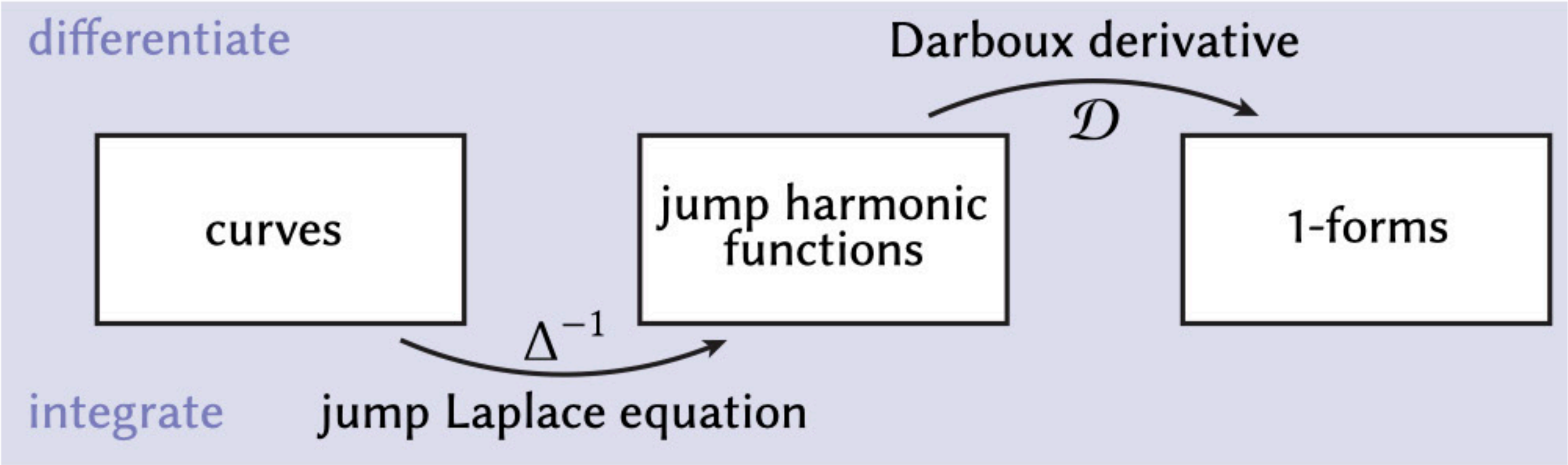
Completing the round-trip

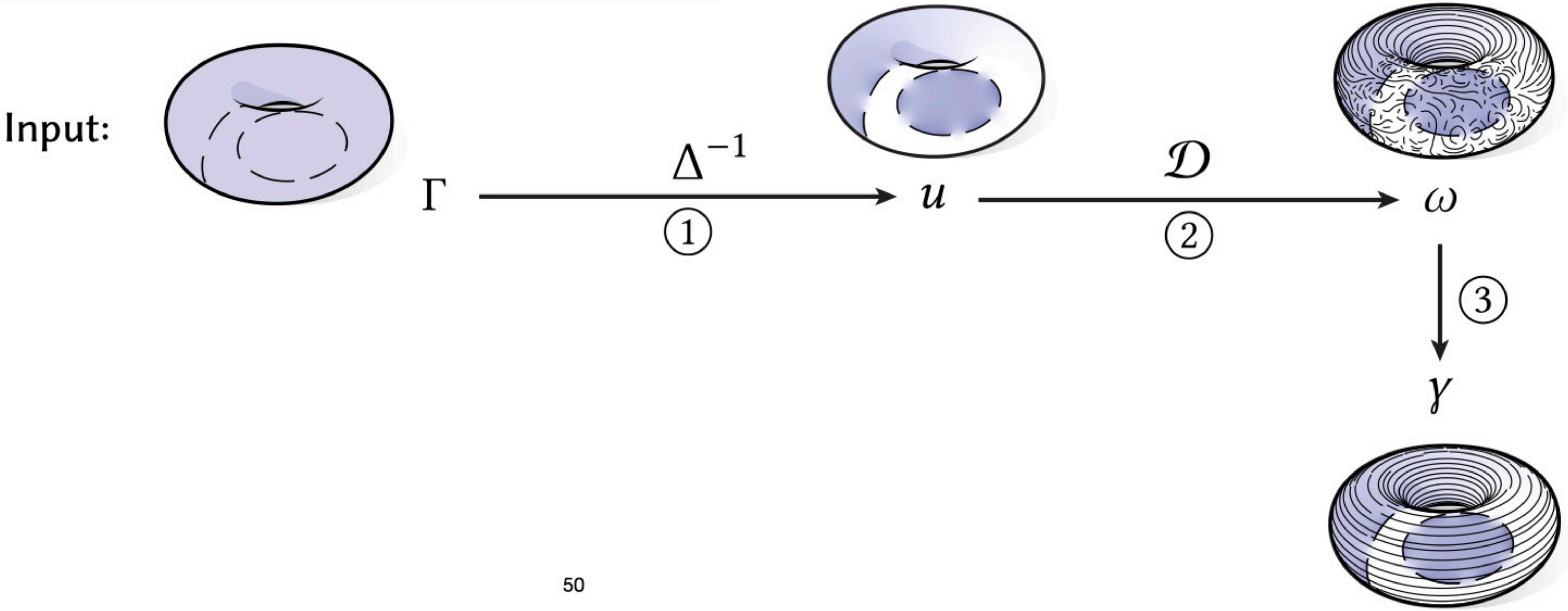


Input:



Completing the round-trip





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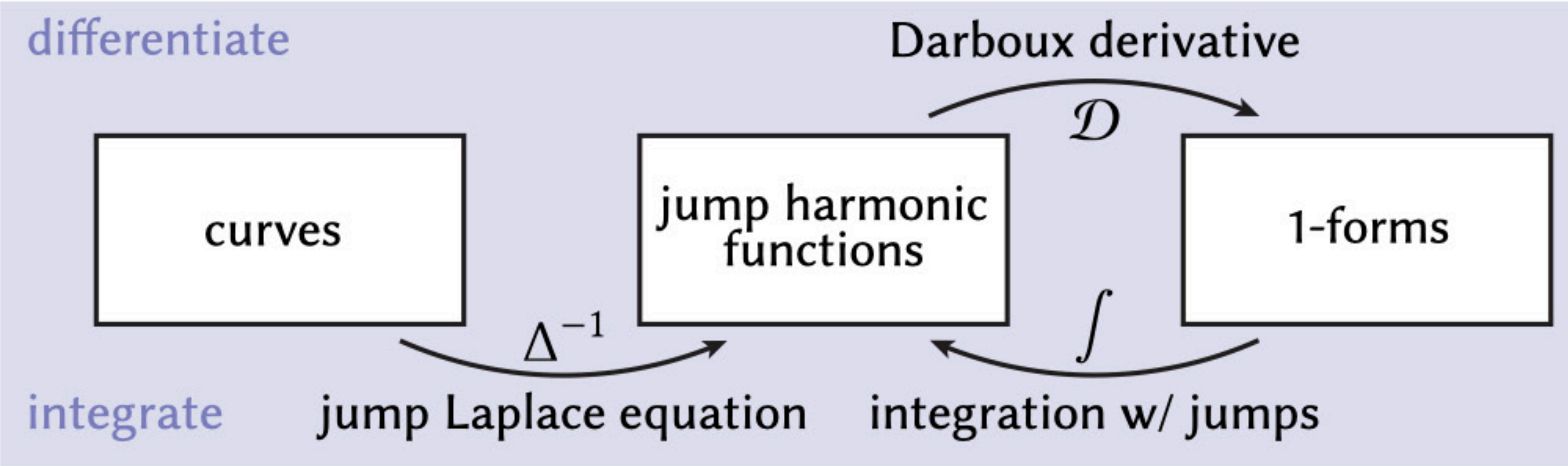

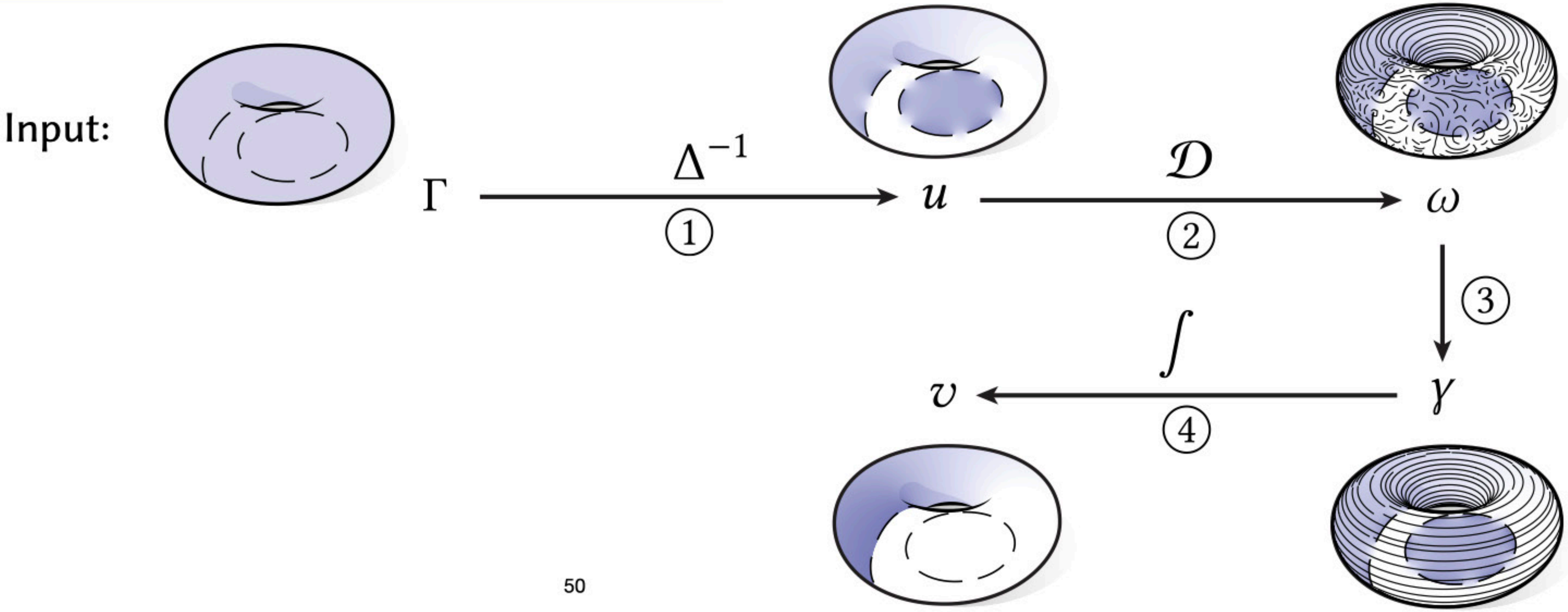

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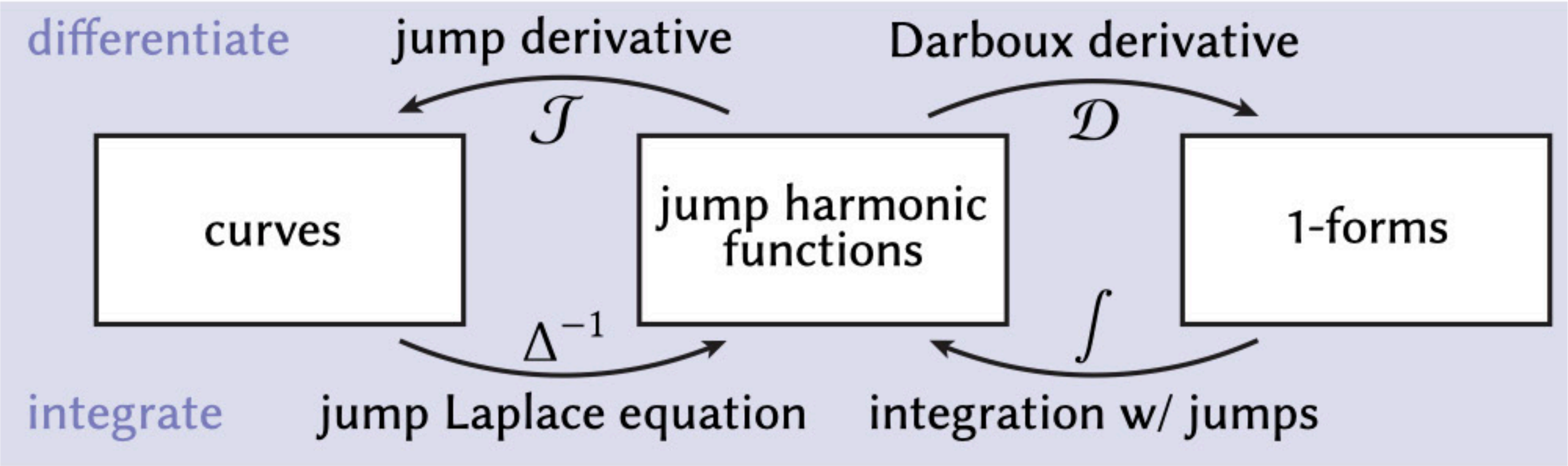
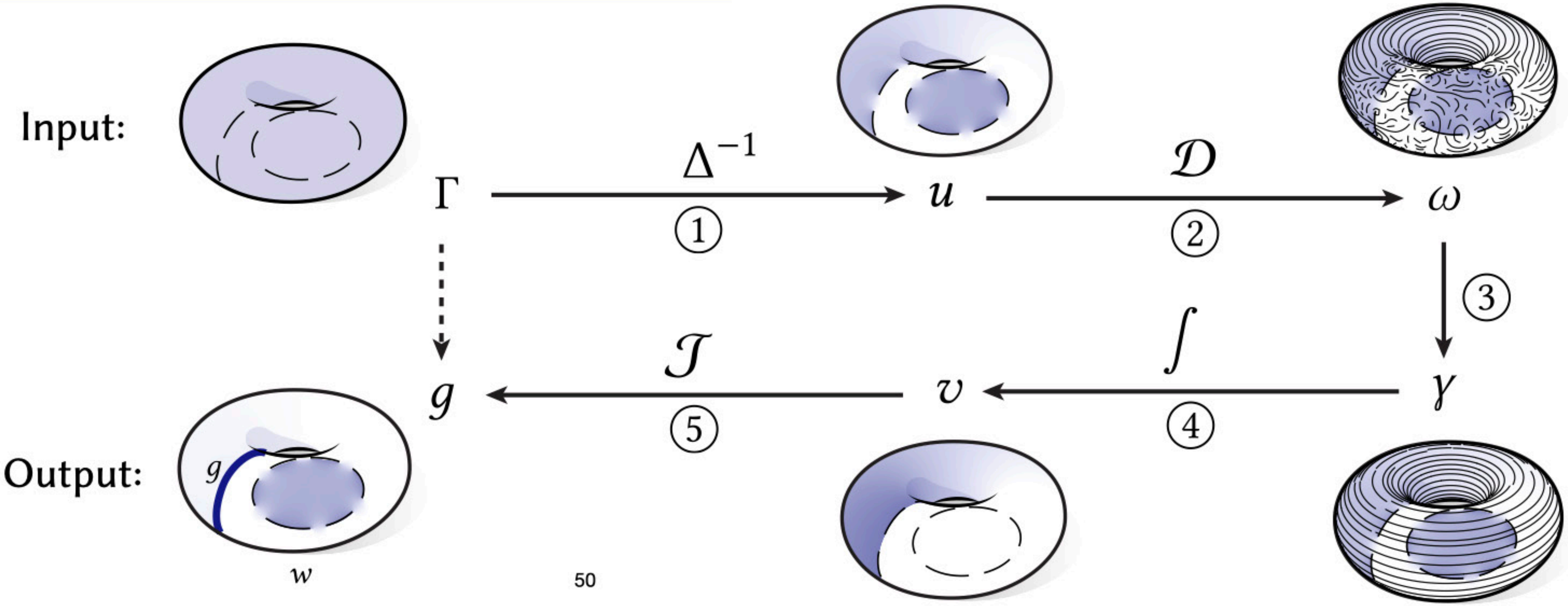
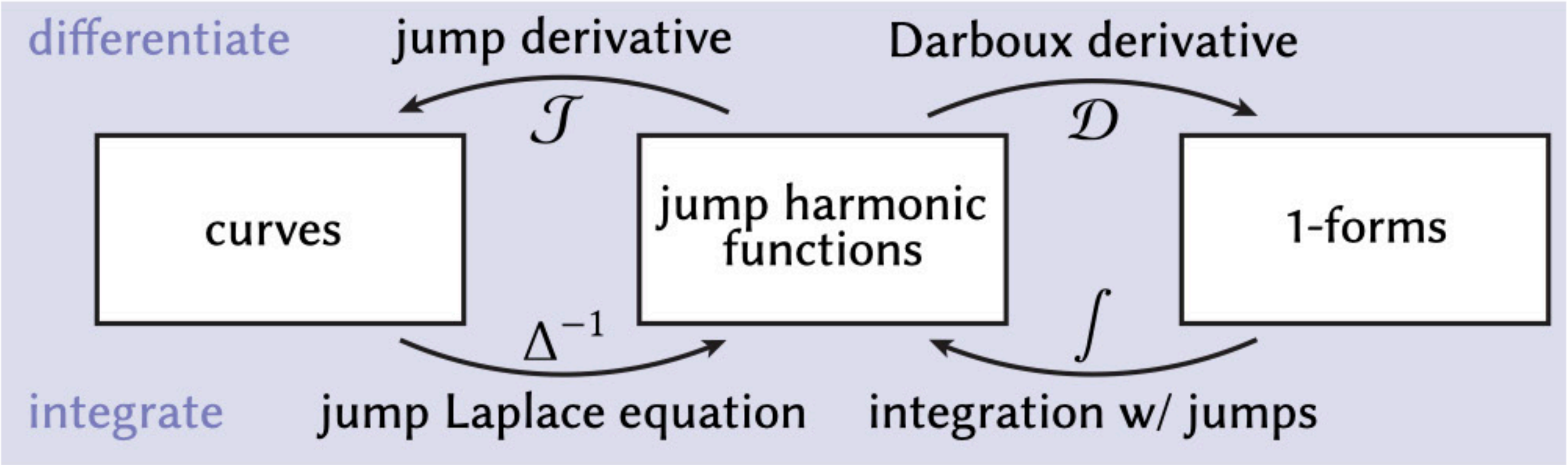
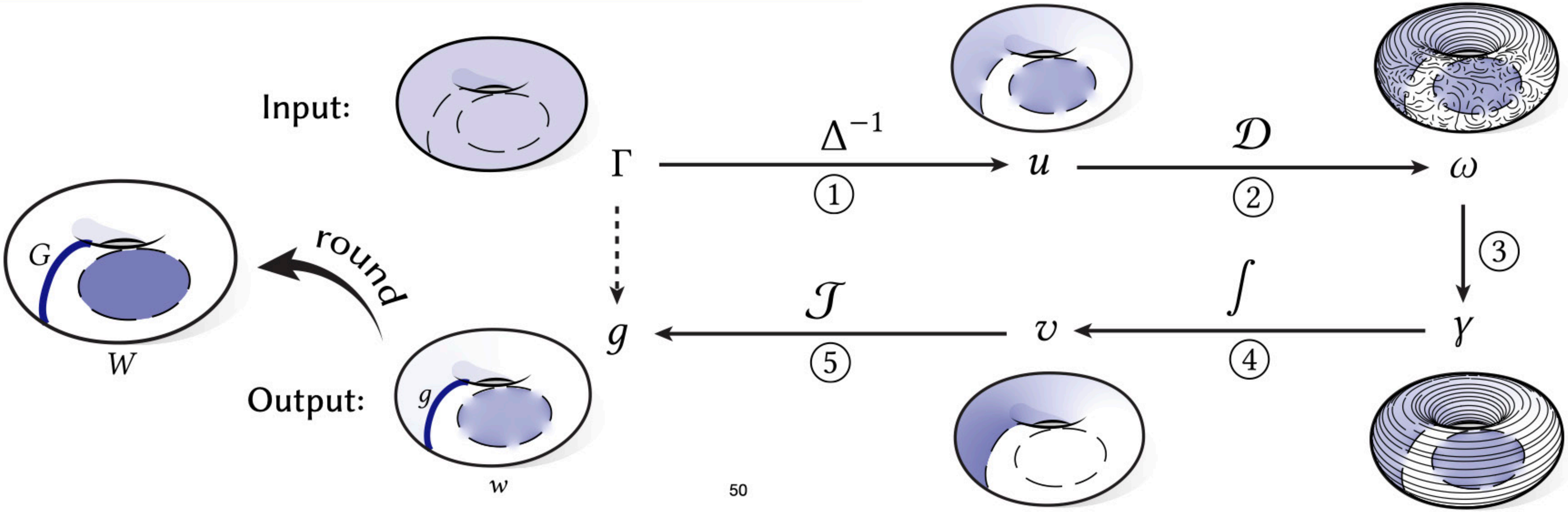




Completing the round-trip





Algorithm summary

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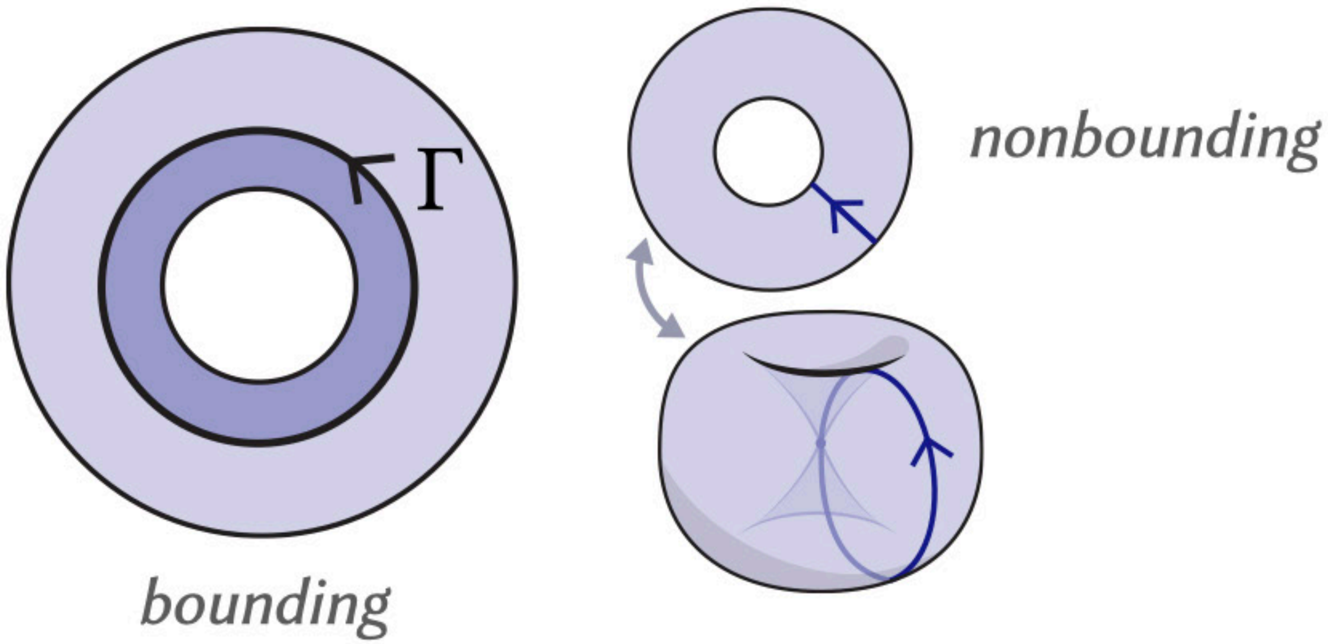
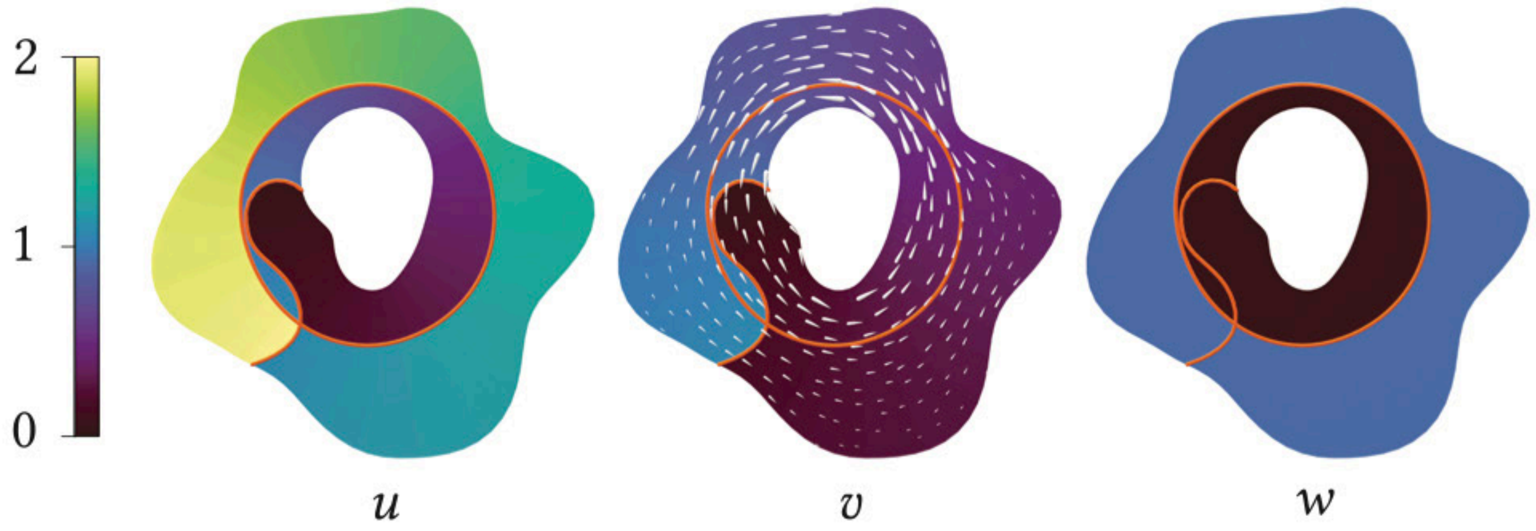
if surface M is multiply-connected

Surfaces with boundary

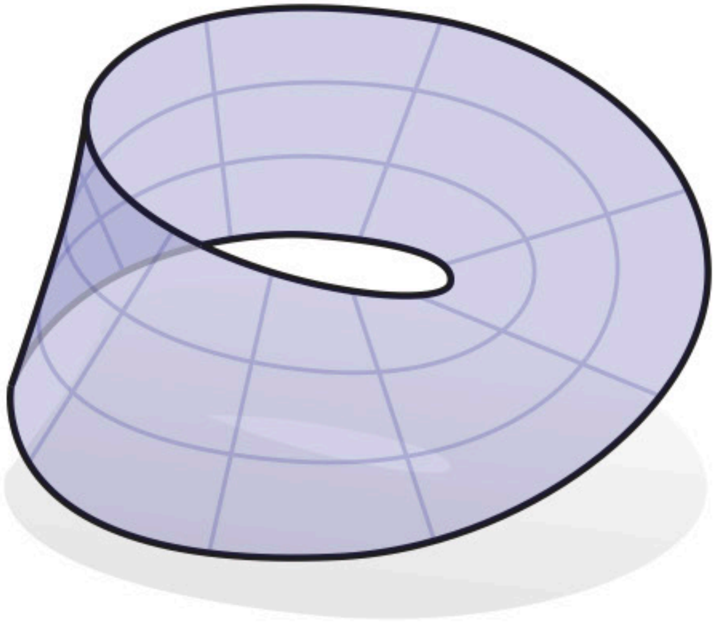
Bounding curves are those congruent to zero in the *relative homology* group $H_1(M, \partial M)$.



The rest of the theory follows.

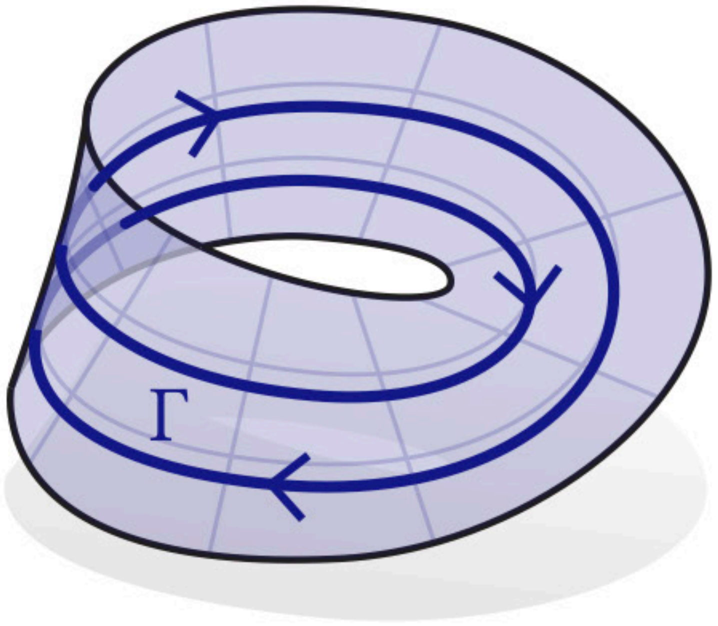


Non-orientable surfaces



Non-orientable surfaces

orientable curve



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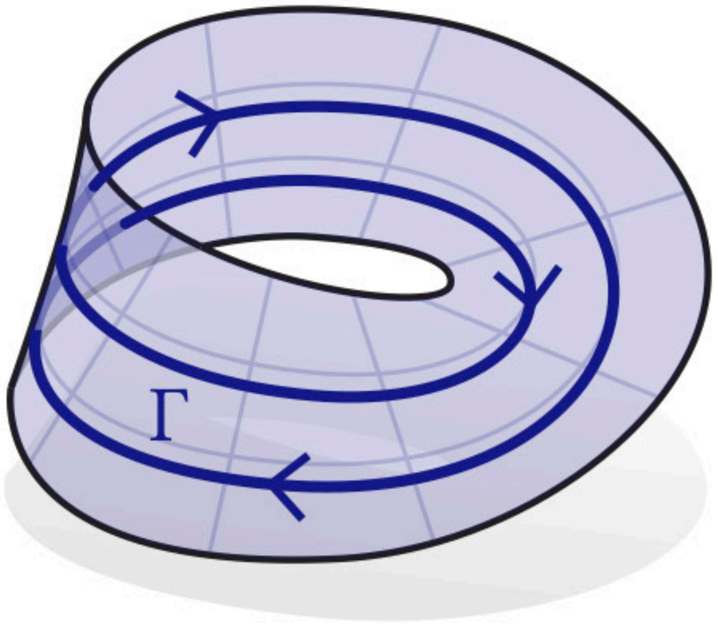

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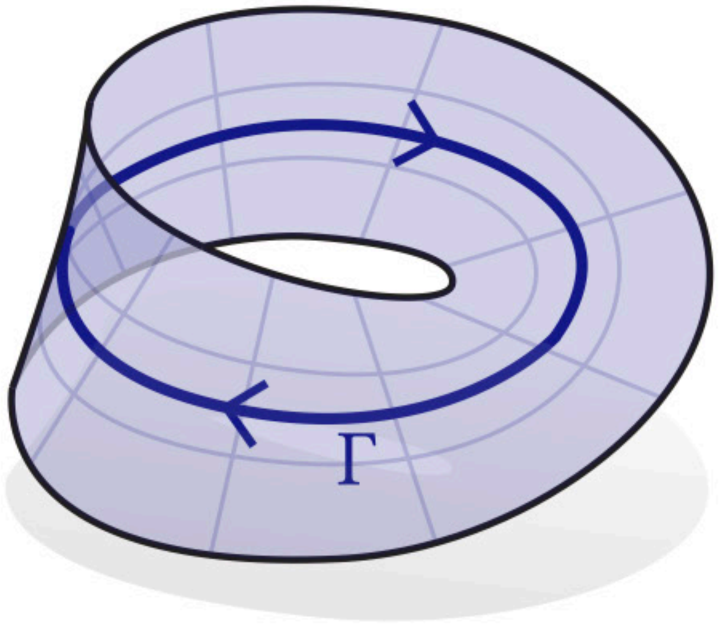

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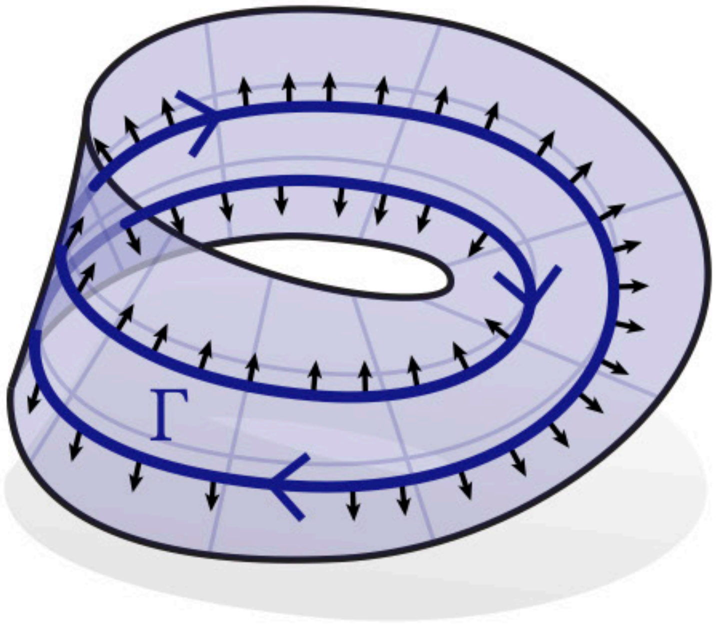
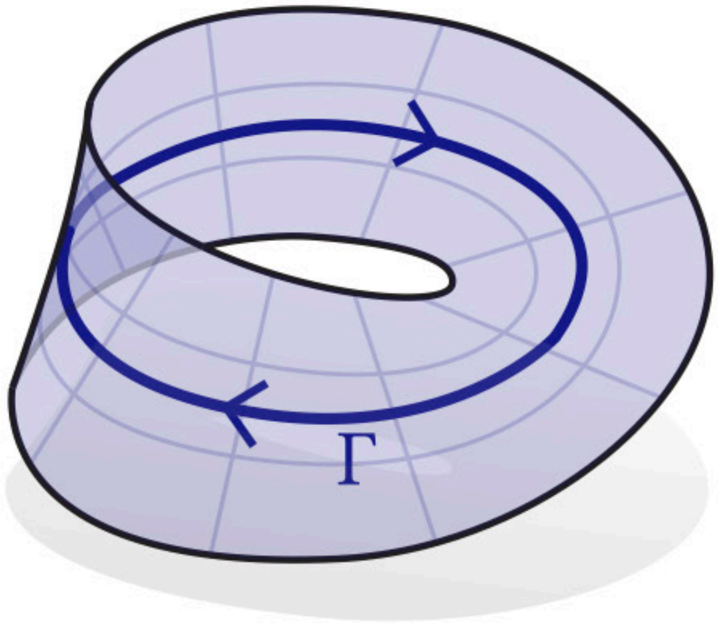


non-orientable curve



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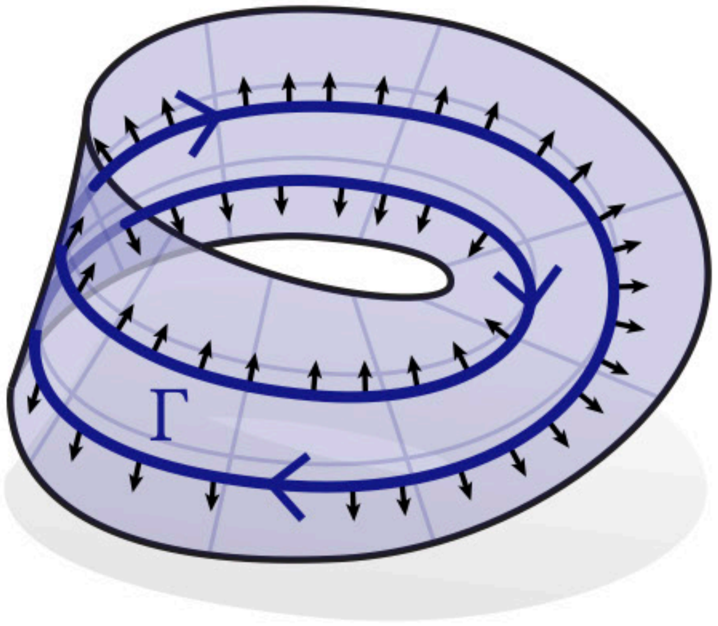

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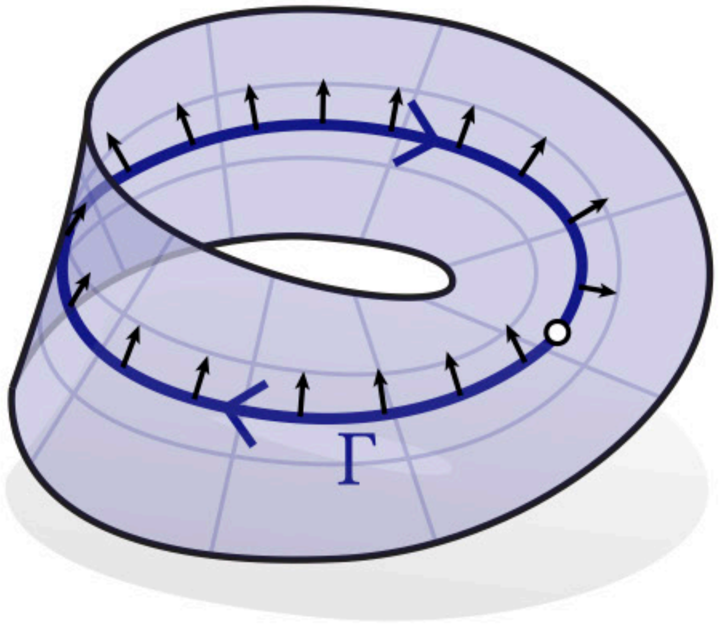

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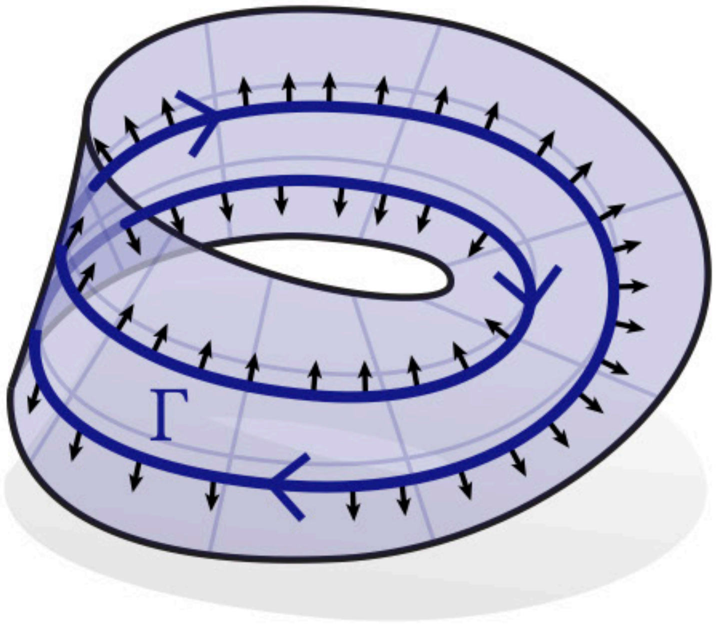

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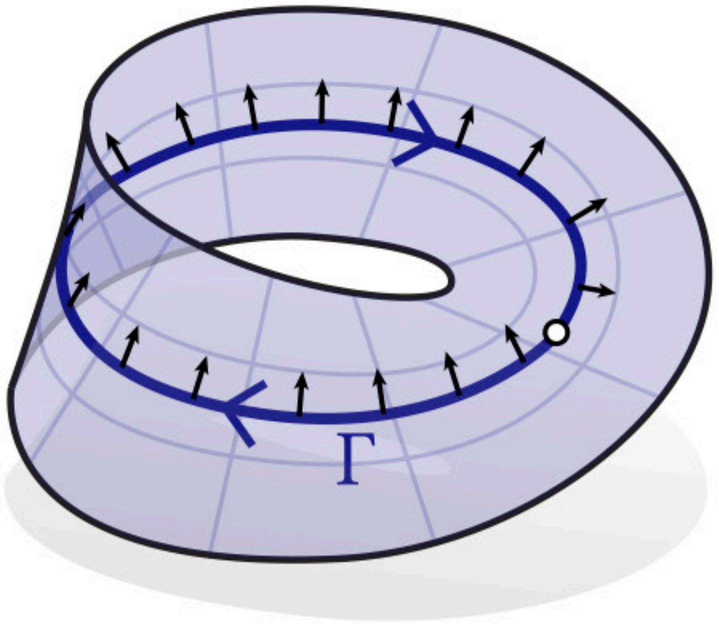
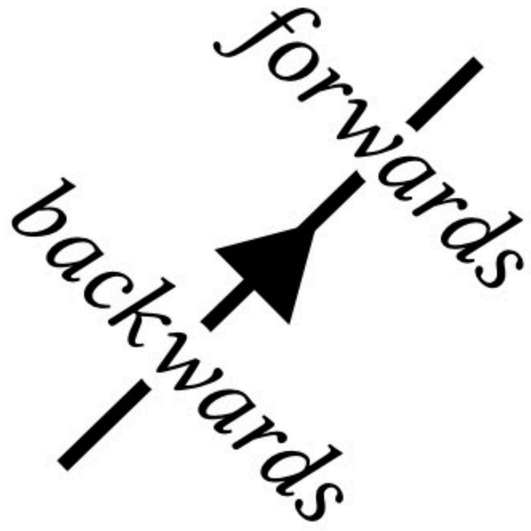
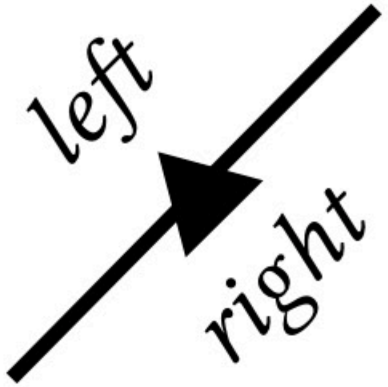


tangential



normal



Non-orientable surfaces

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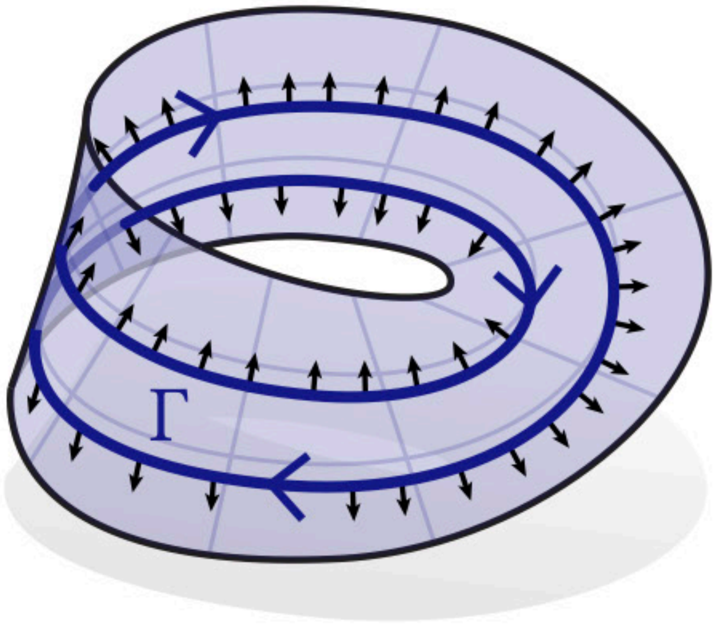

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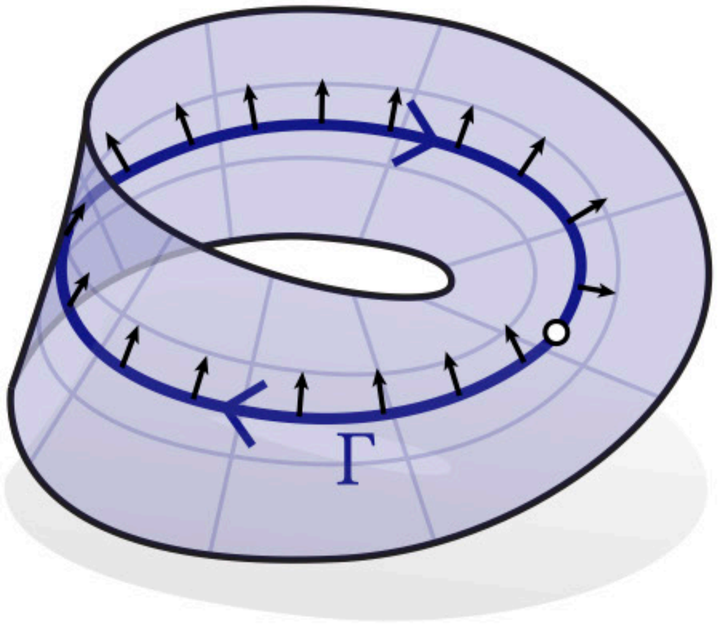

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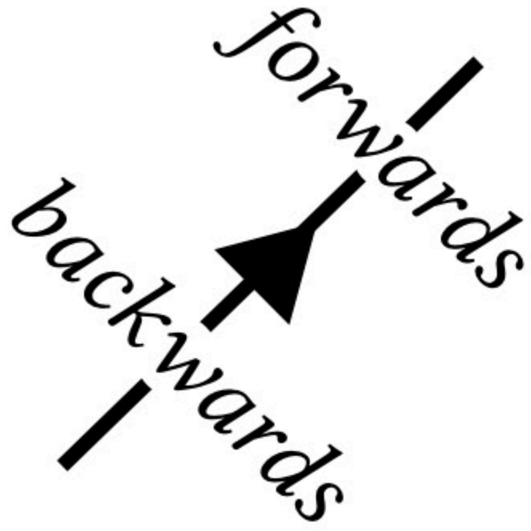

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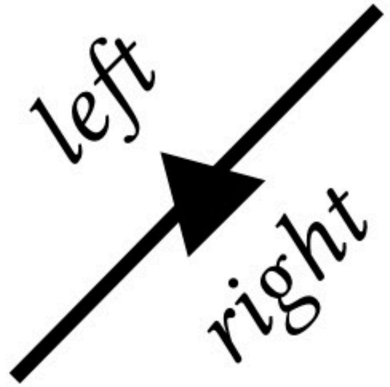
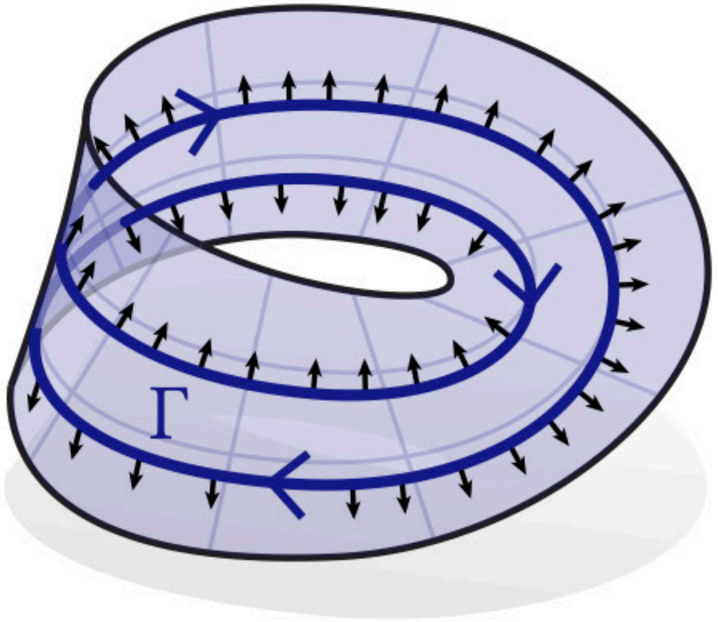


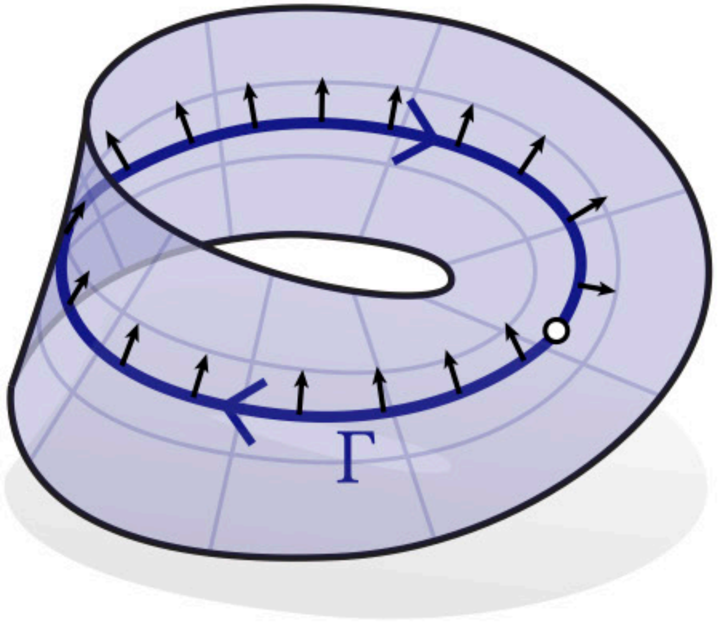
On orientable surfaces,
tangential orientation ≡ normal orientation

Non-orientable surfaces

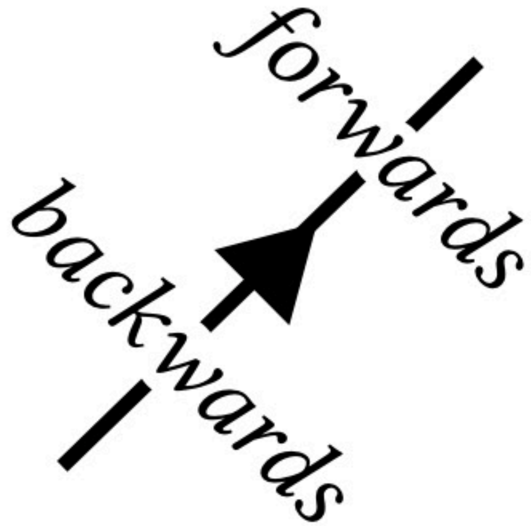
orientable curve



non-orientable curve



tangential



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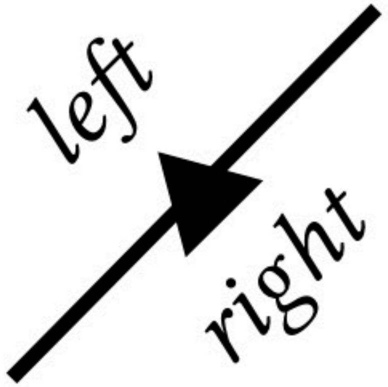

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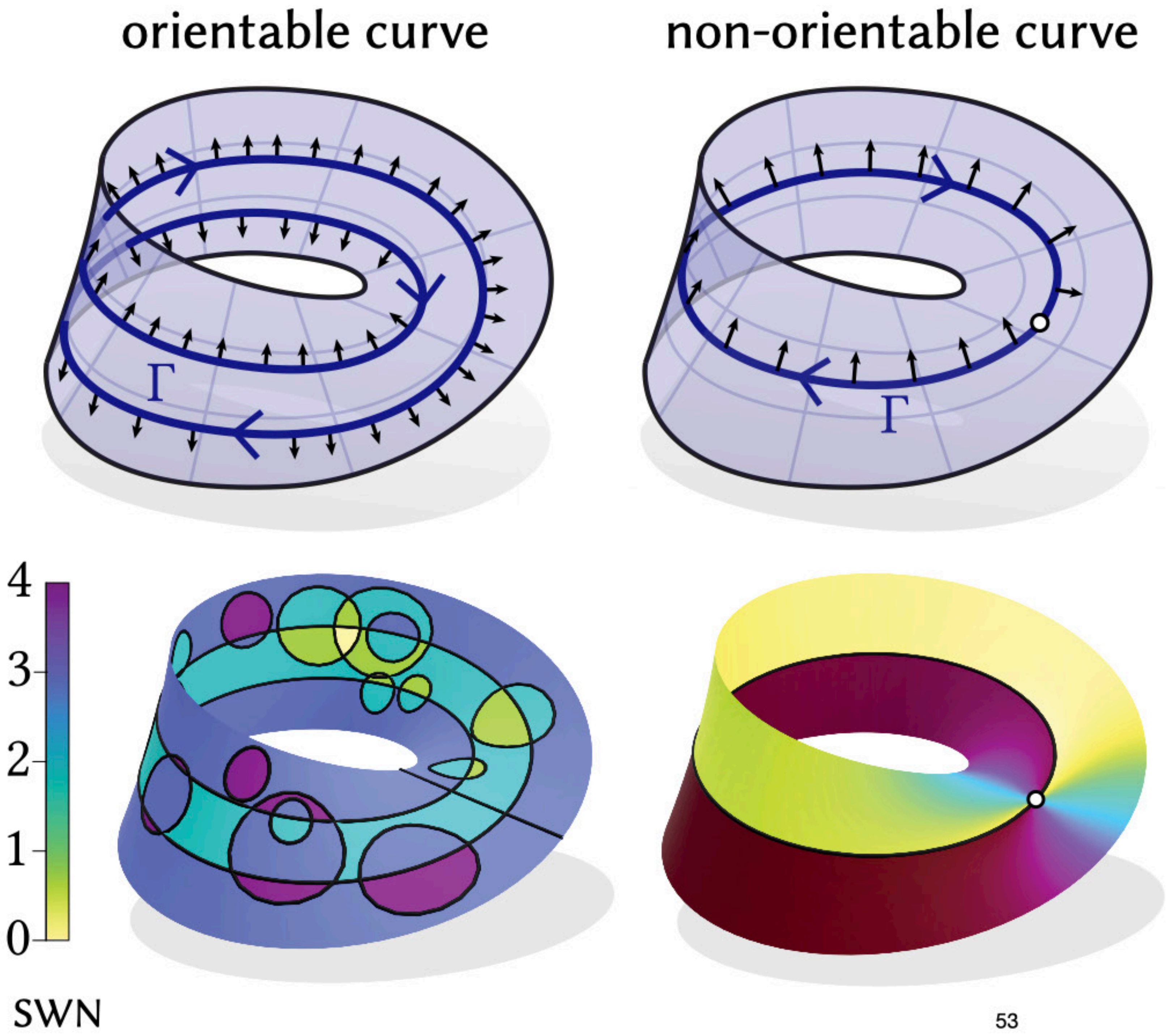
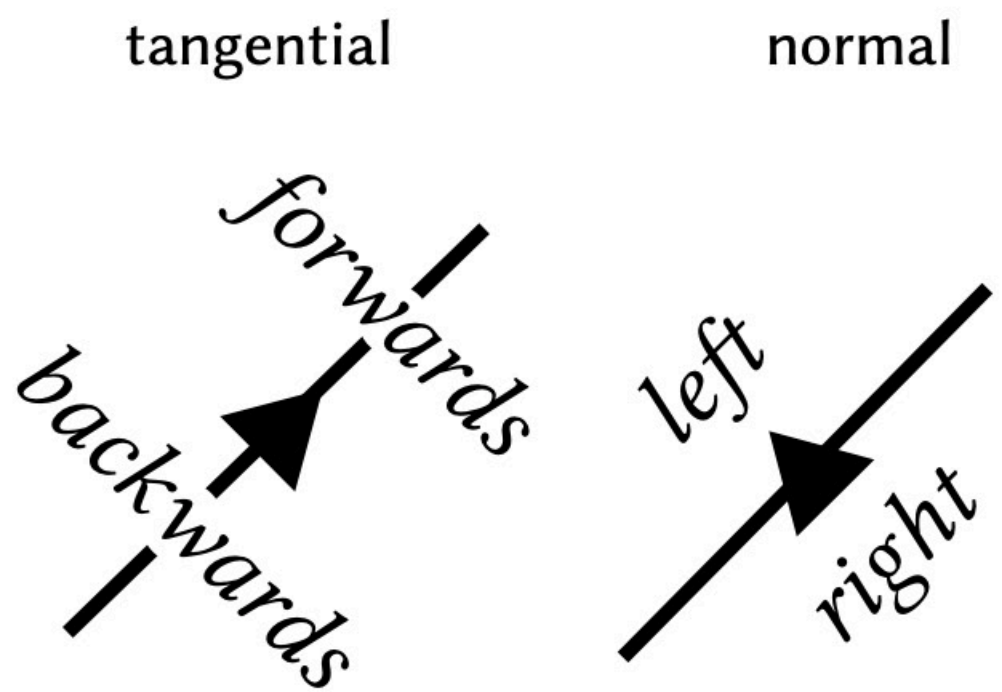
On **non**-orientable surfaces,
must specify normal orientation

Non-orientable surfaces





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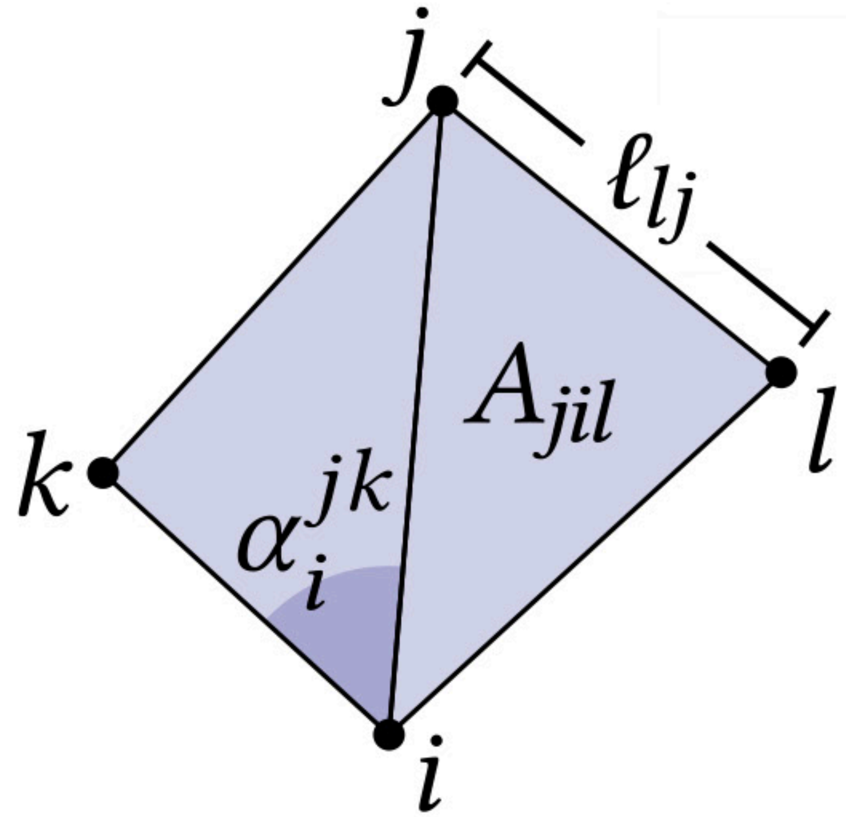
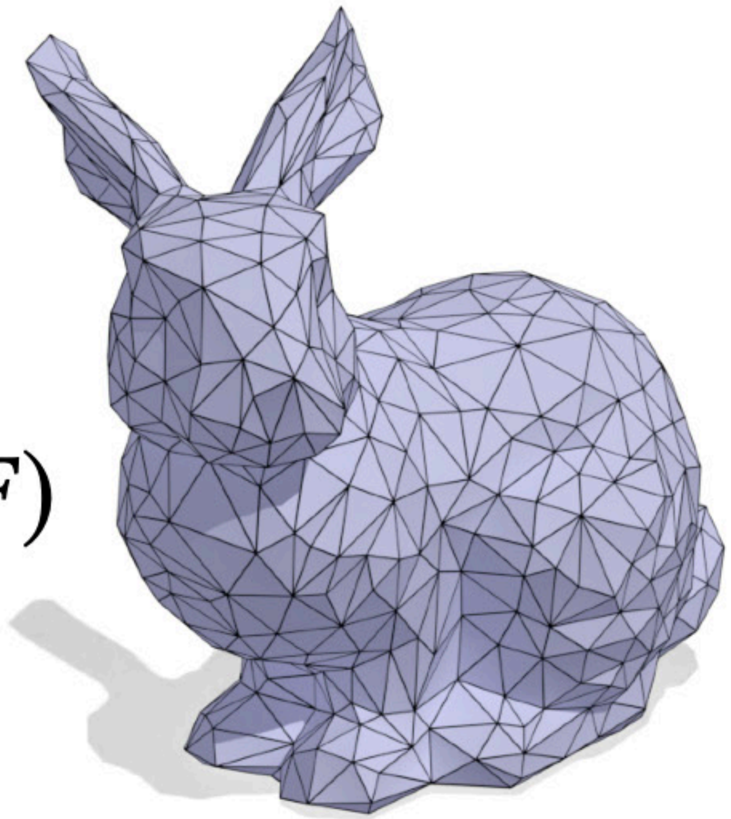
DISCRETIZATION

Curves & regions



$M = (V, E, F)$



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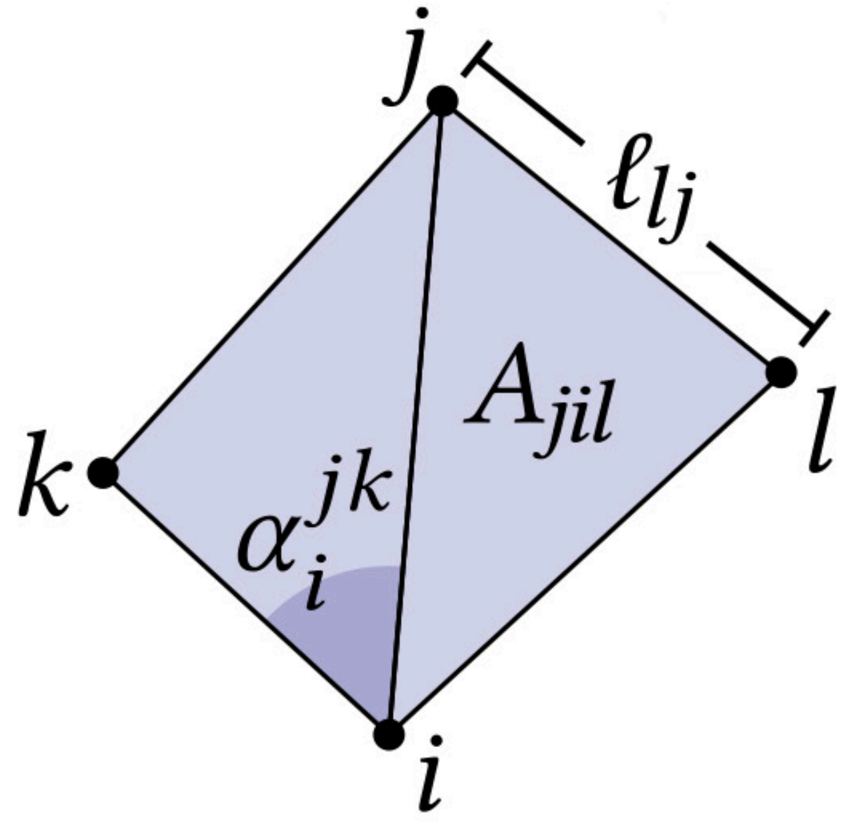
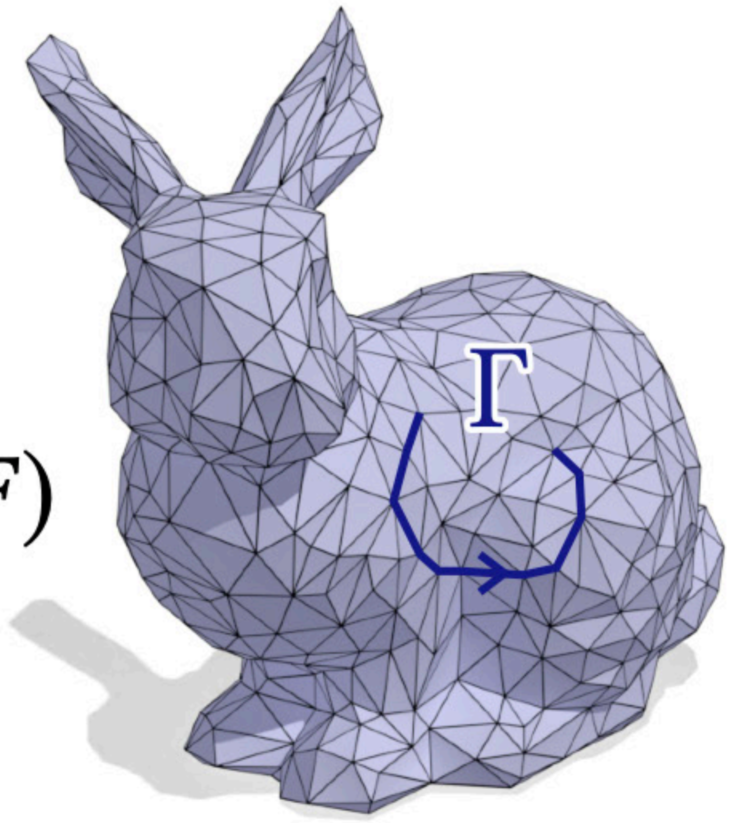
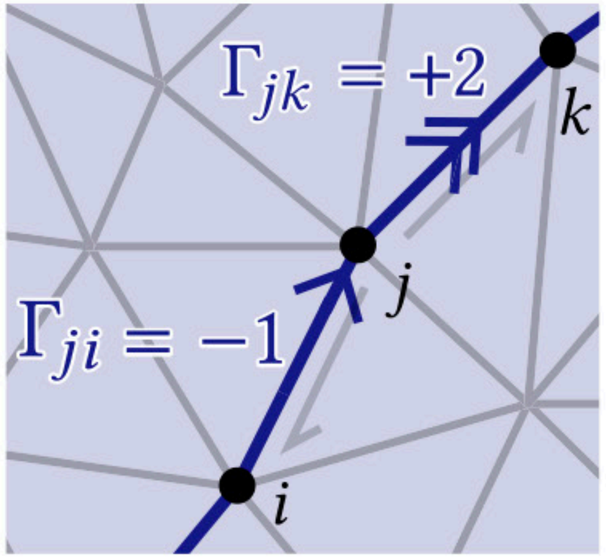


Γ is a **1-chain,** i.e. a signed integer per edge.

Regions are **2-chains,** signed integers per face.

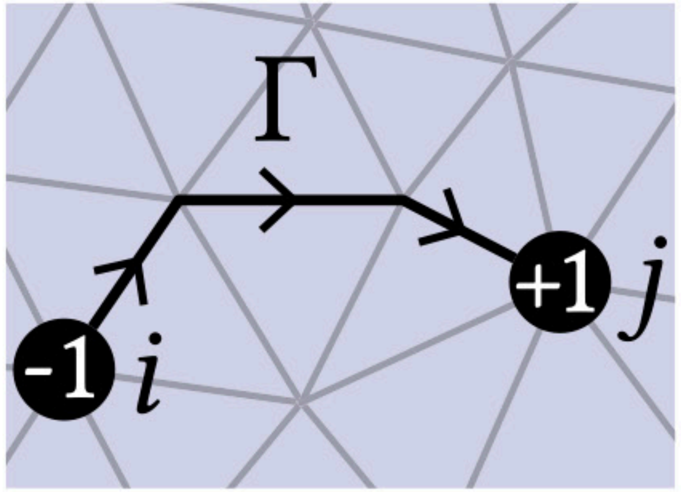
$$M = (V, E, F)$$





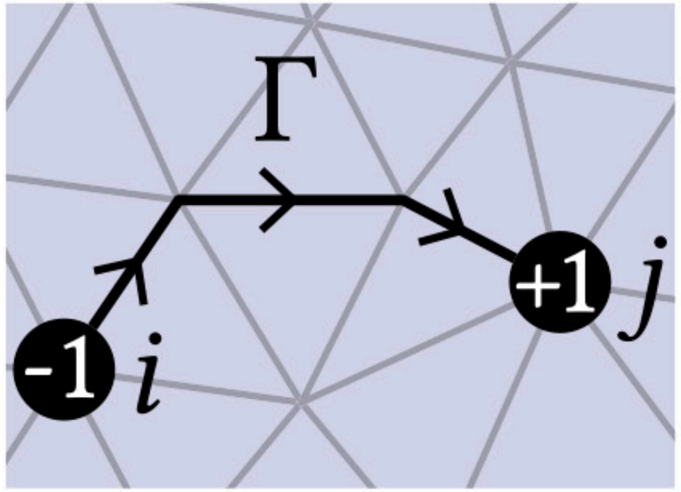
Endpoints

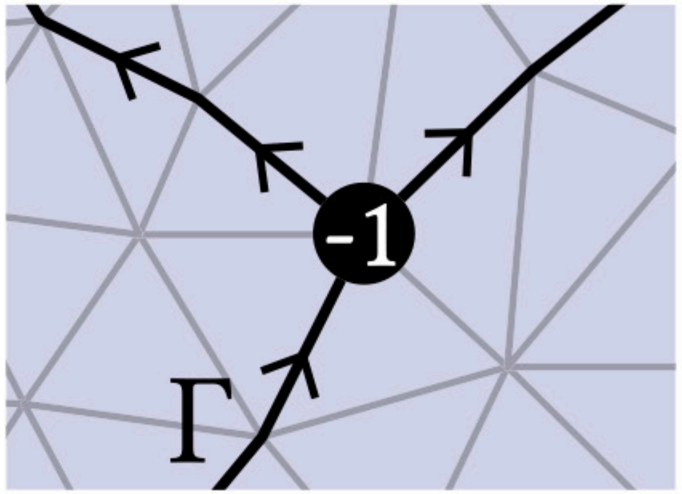
The boundary of Γ is a *0-chain,* $(\partial\Gamma)_i := -\Sigma_{ij}\Gamma_{ij}$



Endpoints

The boundary of Γ is a *0-chain,* $(\partial\Gamma)_i := -\Sigma_{ij}\Gamma_{ij}$





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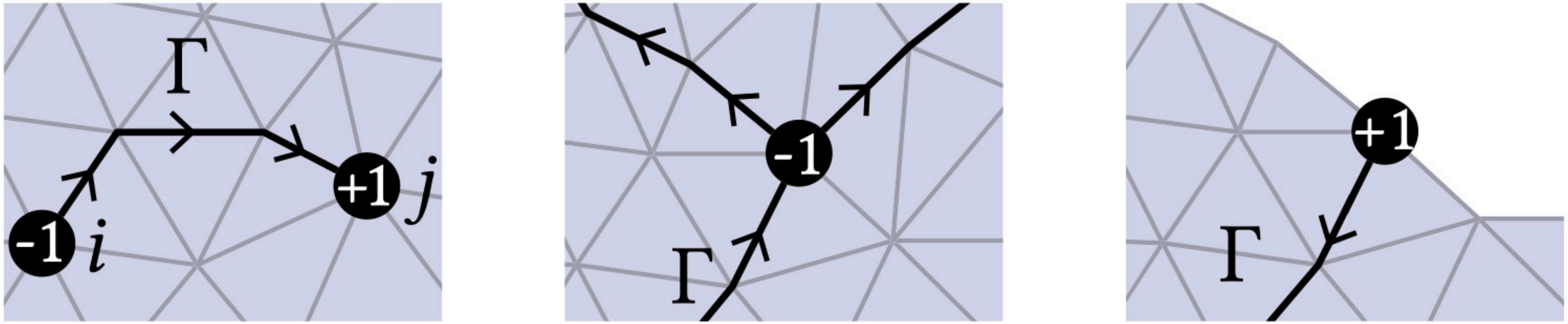

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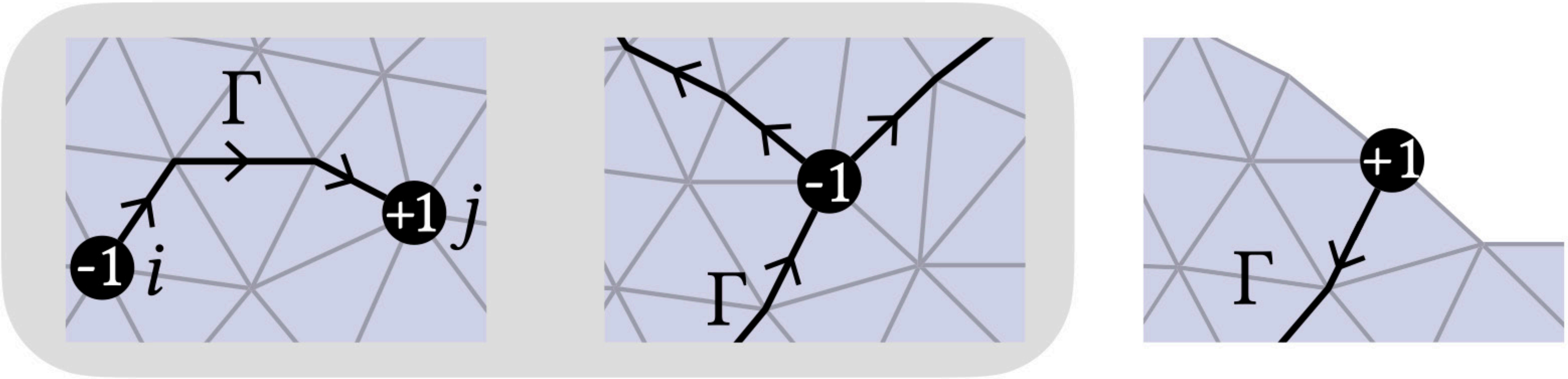

interior endpoints

Endpoints

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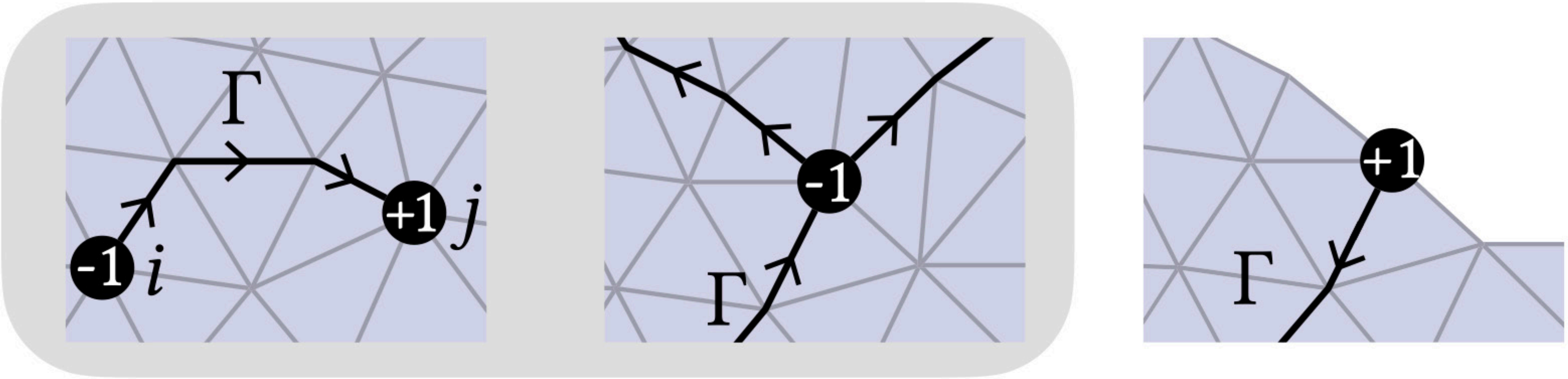
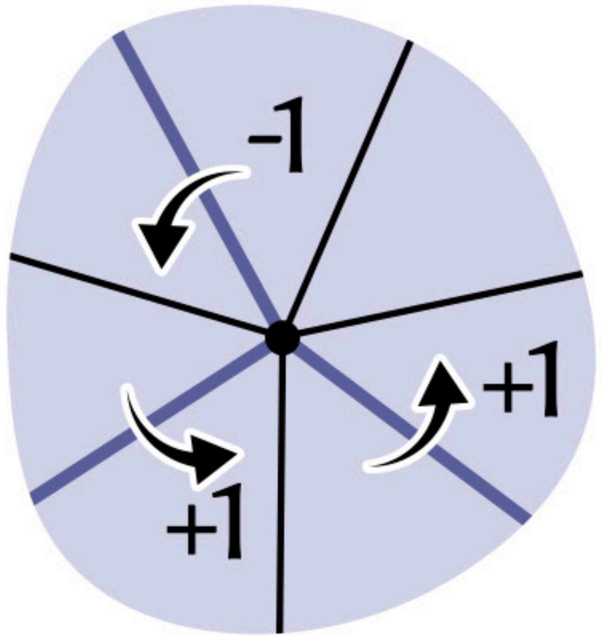


interior endpoints

V^* := set of mesh vertices that are not interior endpoints

E^* := set of edges with both points in V^*

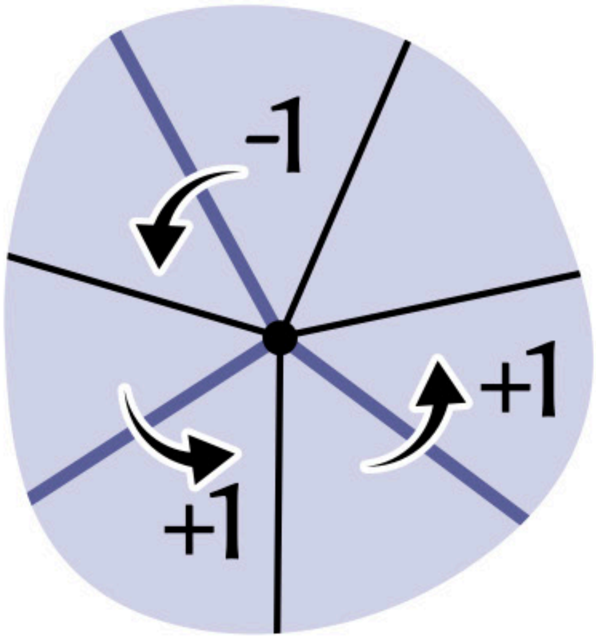
Endpoints are singular



Endpoints represent *singular points:*

There are no corner values compatible with jumps.

Endpoints are singular



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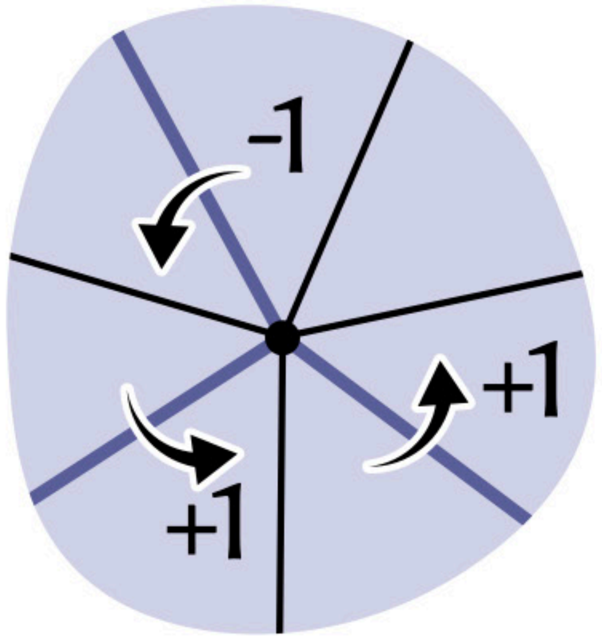

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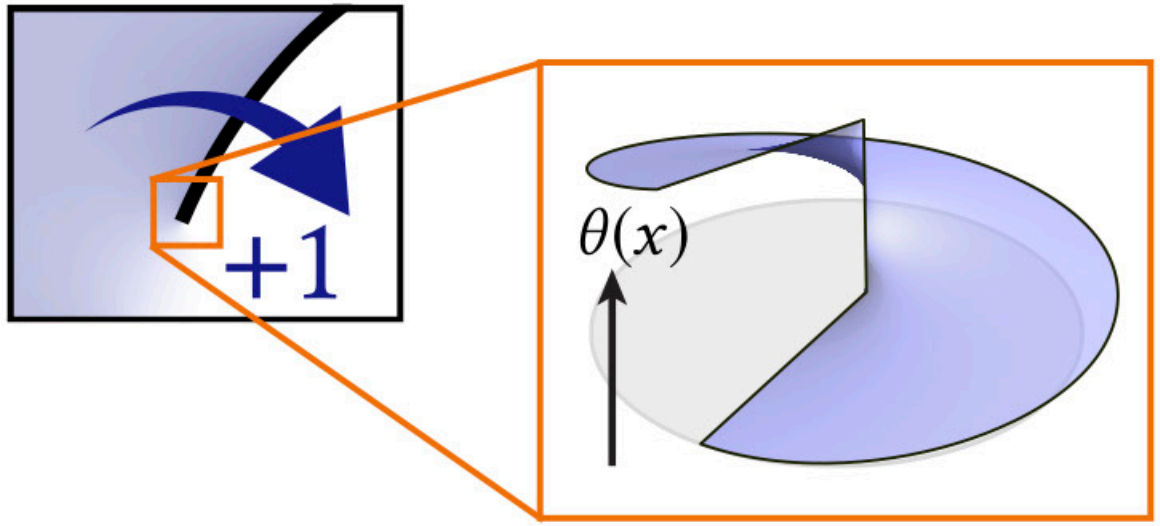
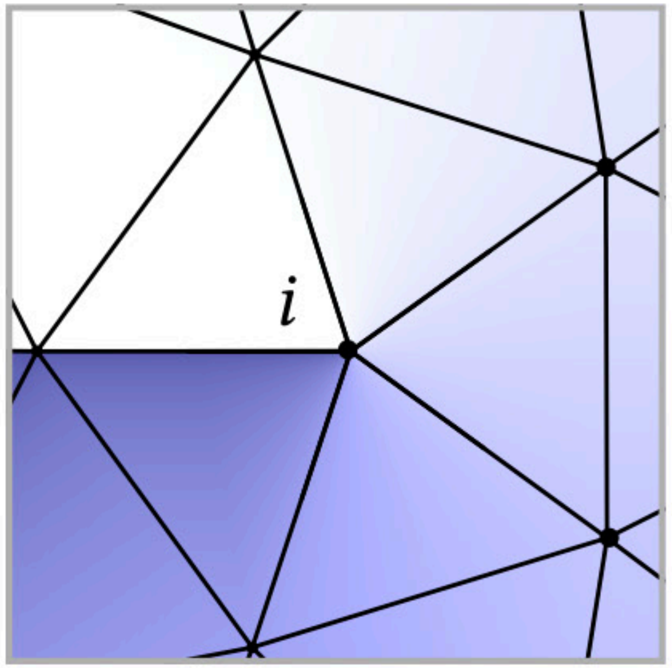
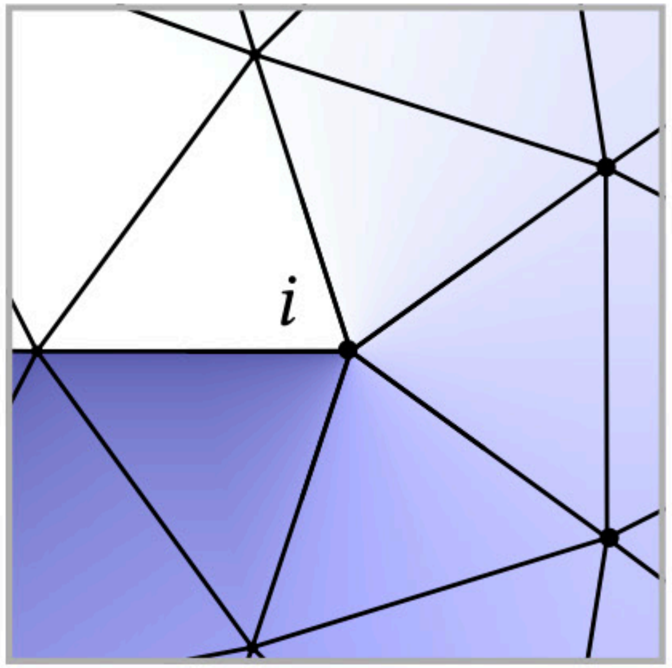
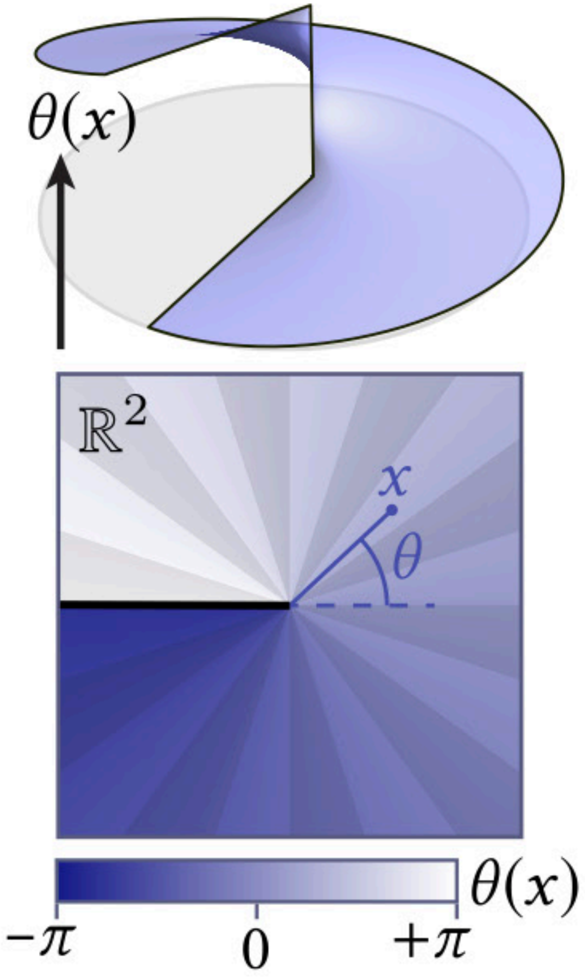


Singular point interpolation

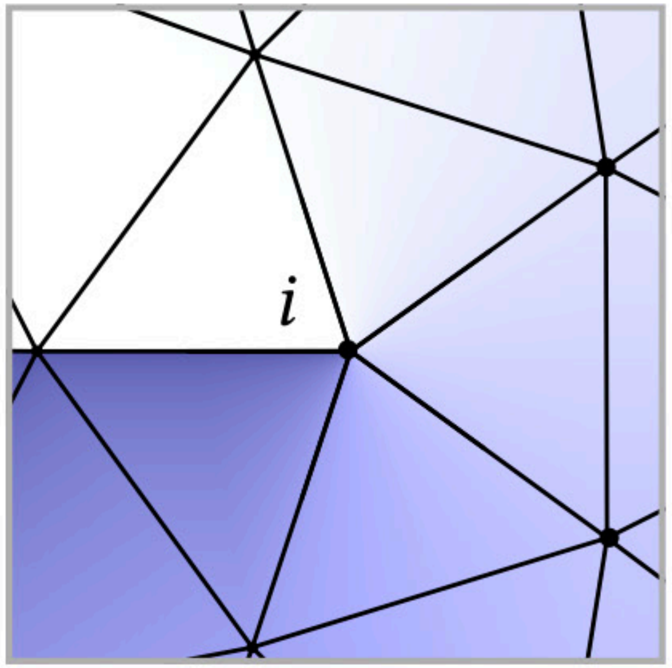


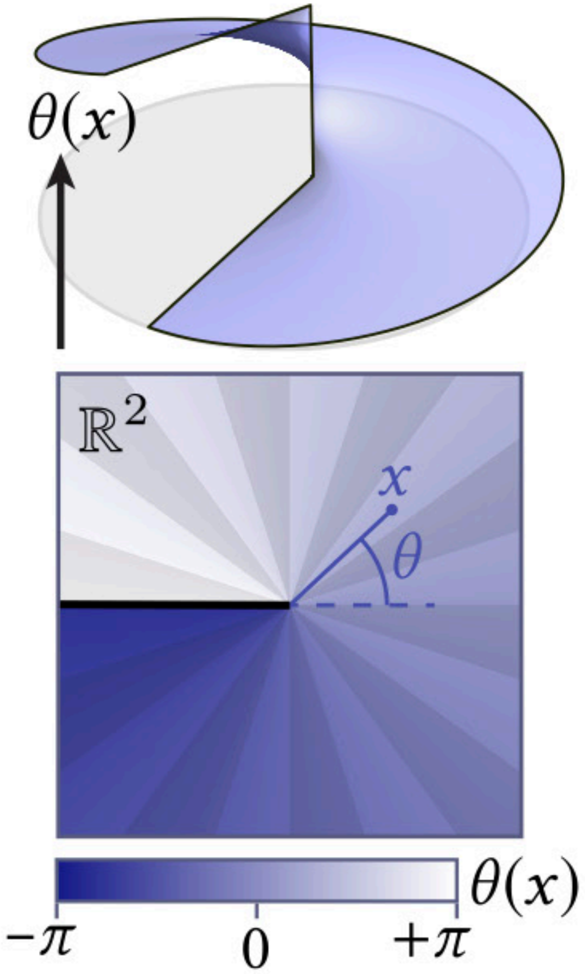
Singular point interpolation

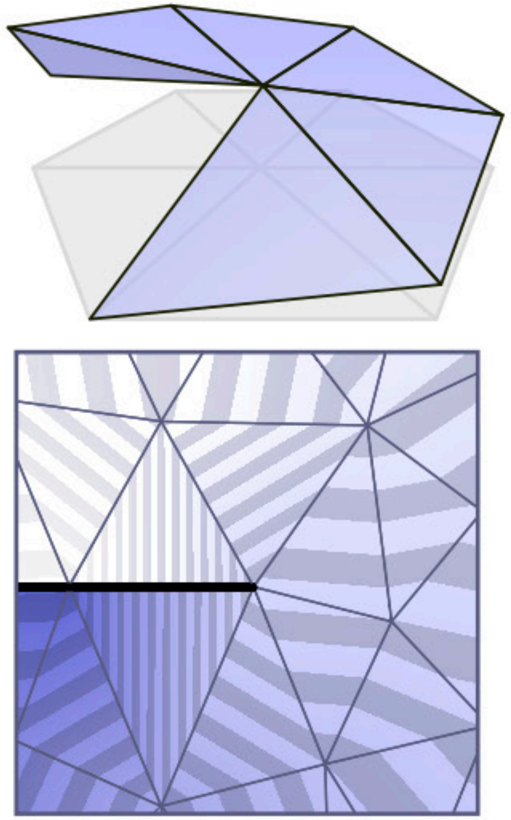




Singular point interpolation



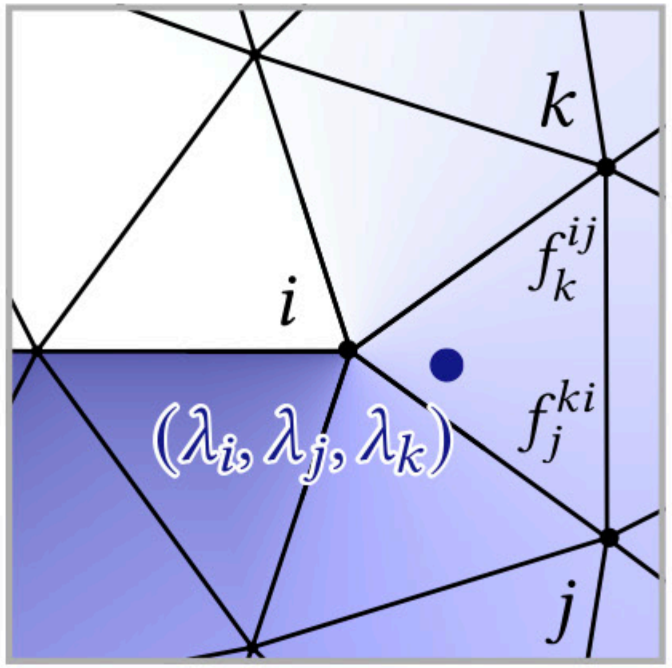


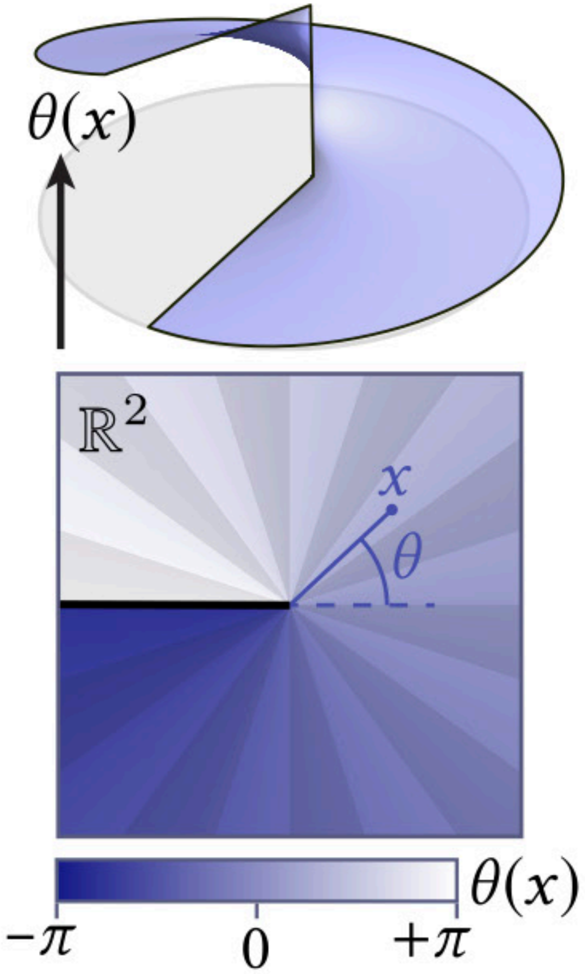


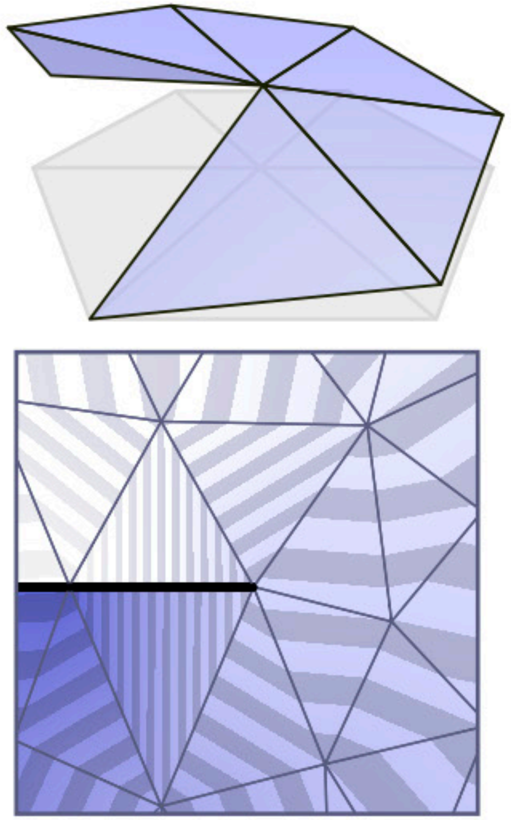
piecewise-linear interpolation of θ

Singular point interpolation

$$f(\lambda_i, \lambda_j, \lambda_k) := \frac{\lambda_j f_j^{ki} + \lambda_k f_k^{ij}}{\lambda_j + \lambda_k}$$







piecewise-linear interpolation of θ

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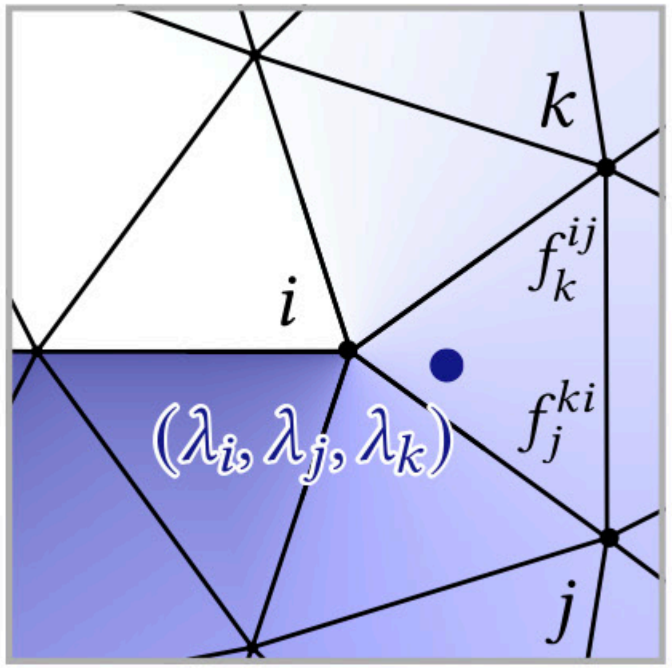
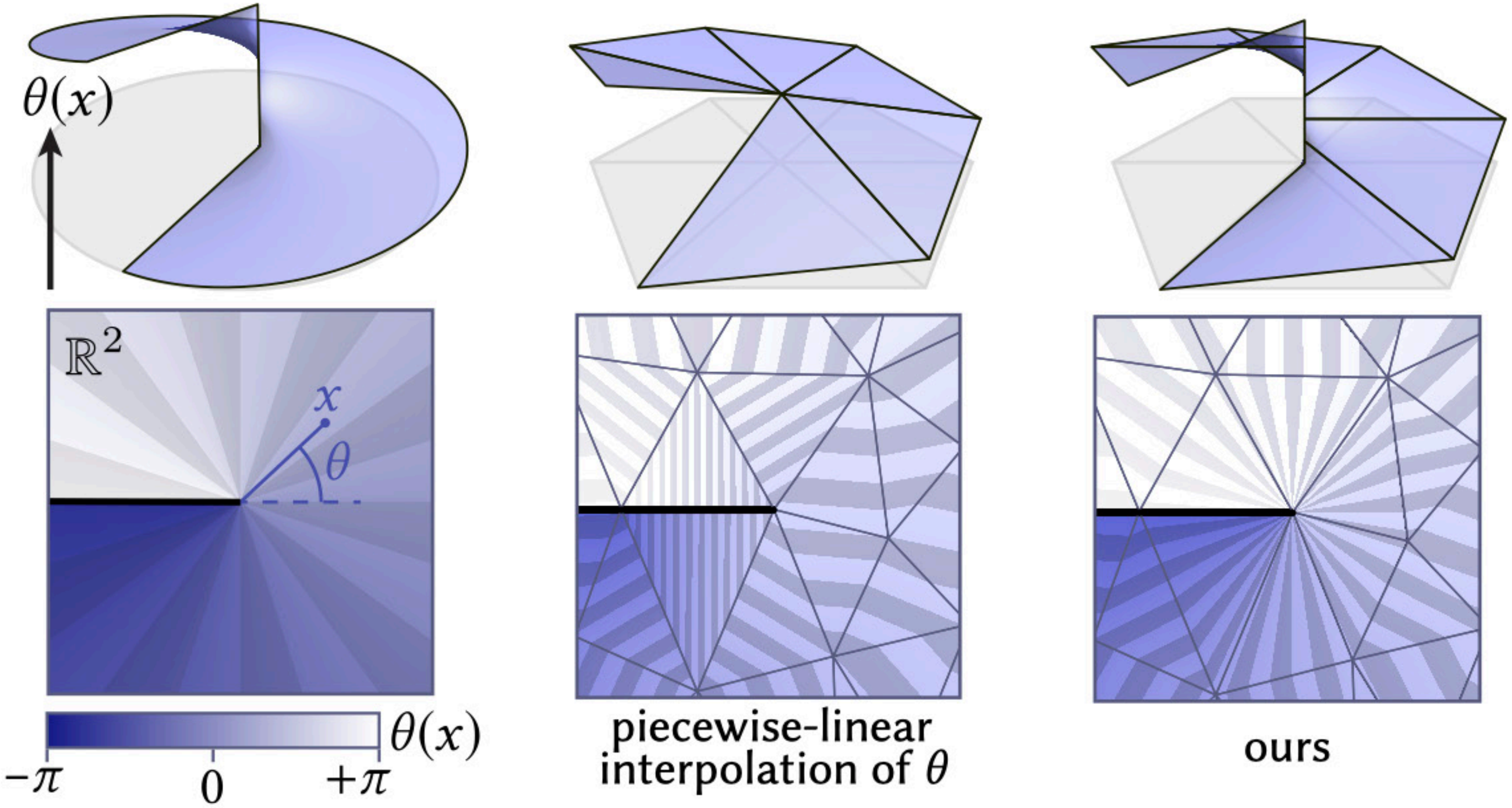
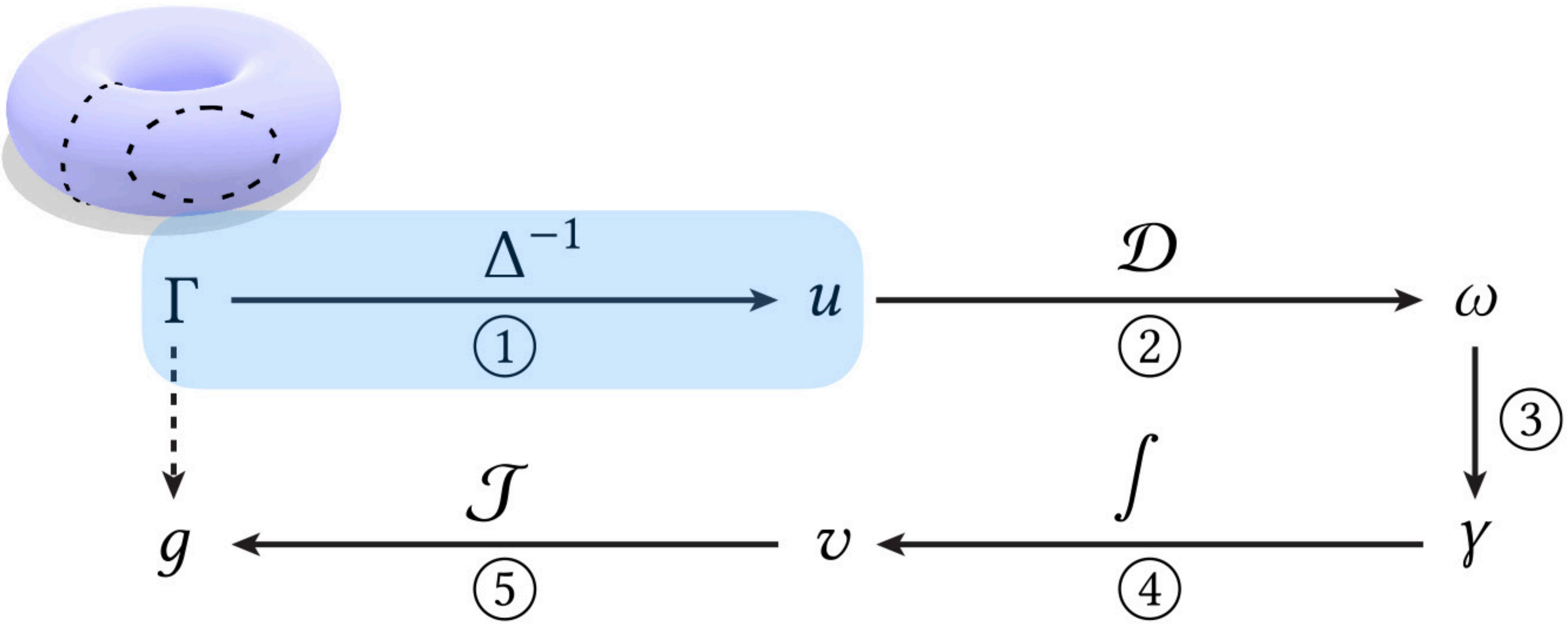
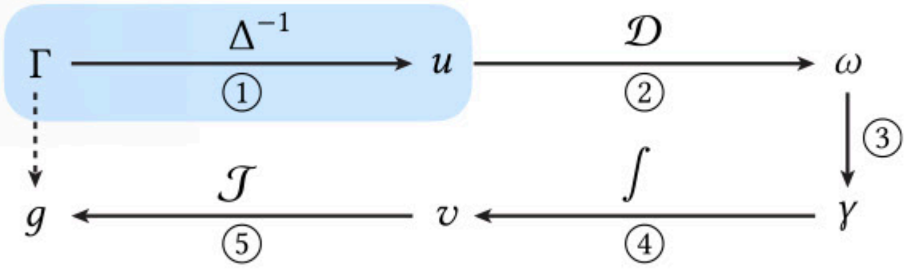




Solving the jump Laplace equation...



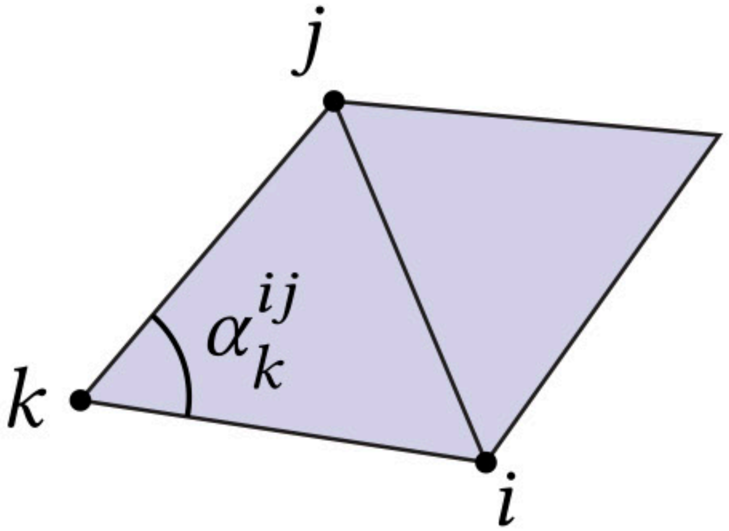
The discrete jump Laplacian



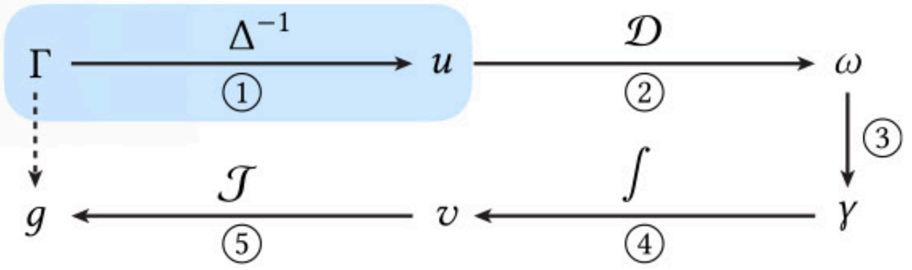
Build the standard cotan Laplacian on V^*:

$$
\begin{aligned}
L_{ij} &= L_{ji} = -\mathrm{w}_{ij}, && \forall ij \in E^* \\
L_{ii} &= \textstyle\sum_{\in E^*} \mathrm{w}_{ij}, && \forall i \in V^*
\end{aligned}
$$

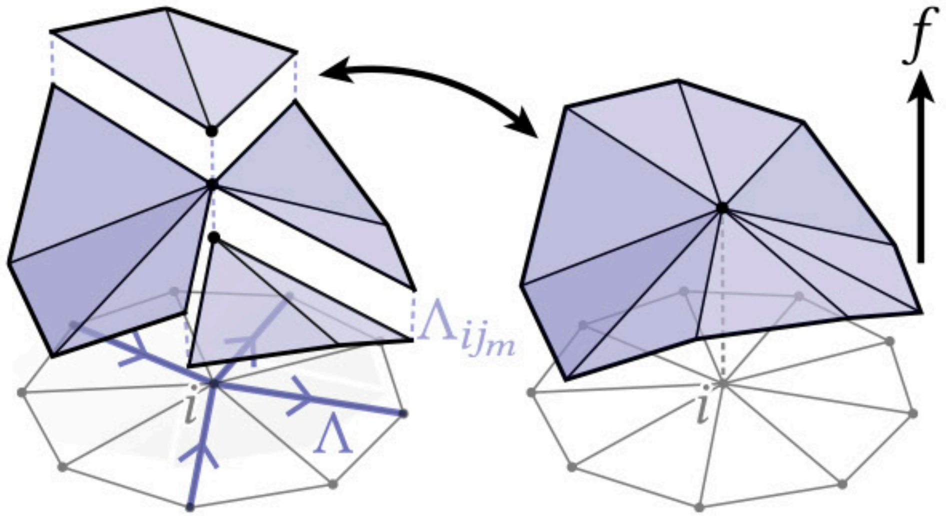
$$\mathrm{w}_{ij} := \frac{1}{2} \sum_{ijk \in F} \cot \alpha_k^{ij}$$



Reduced coordinates



A jump harmonic function f is harmonic "up to jumps" Λ.



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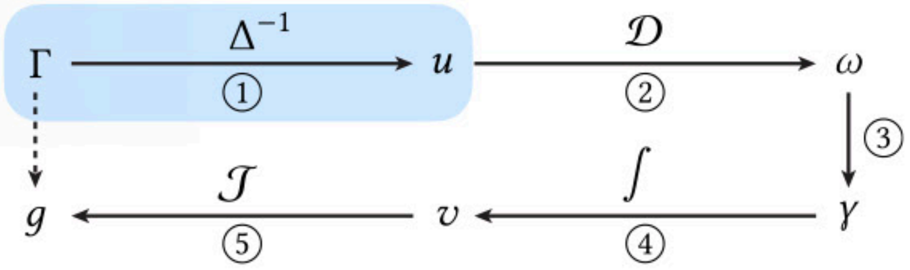

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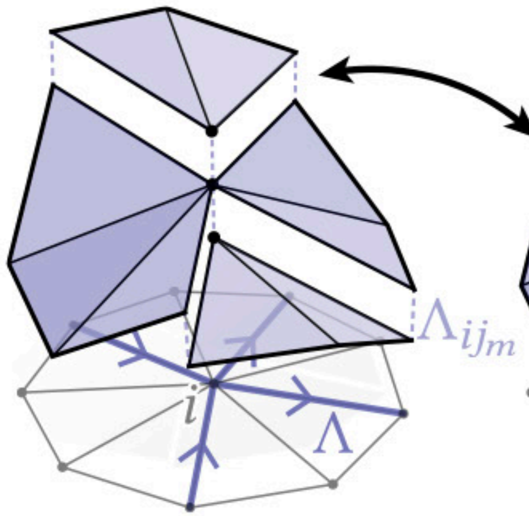
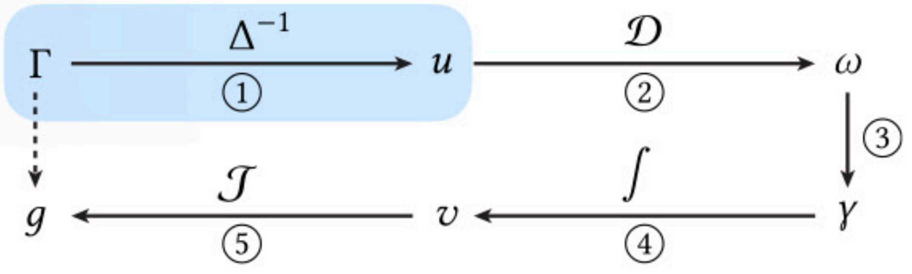


Value per corner →
one value per vertex!

Reduced coordinates



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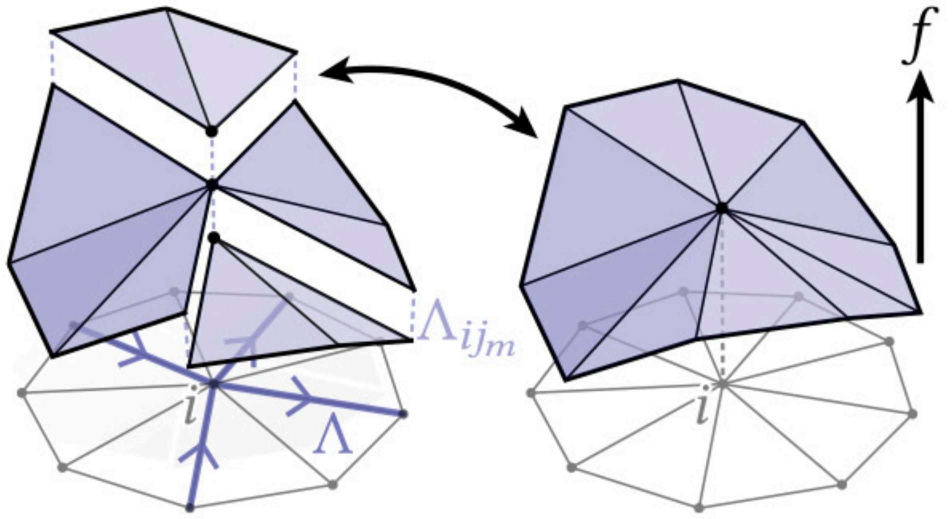
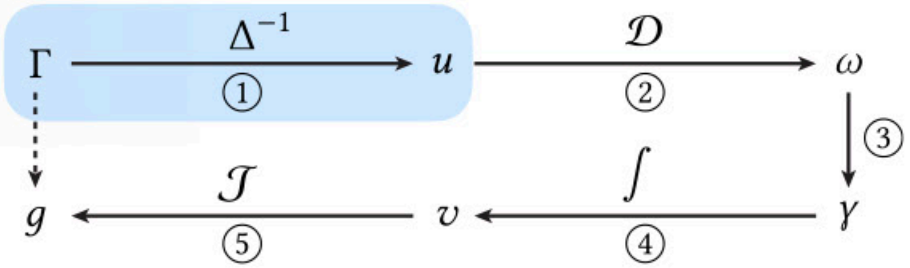


Value per corner →
one value per vertex!

Apply change of variables and solve

$$Lf_0 = b$$

Reduced coordinates



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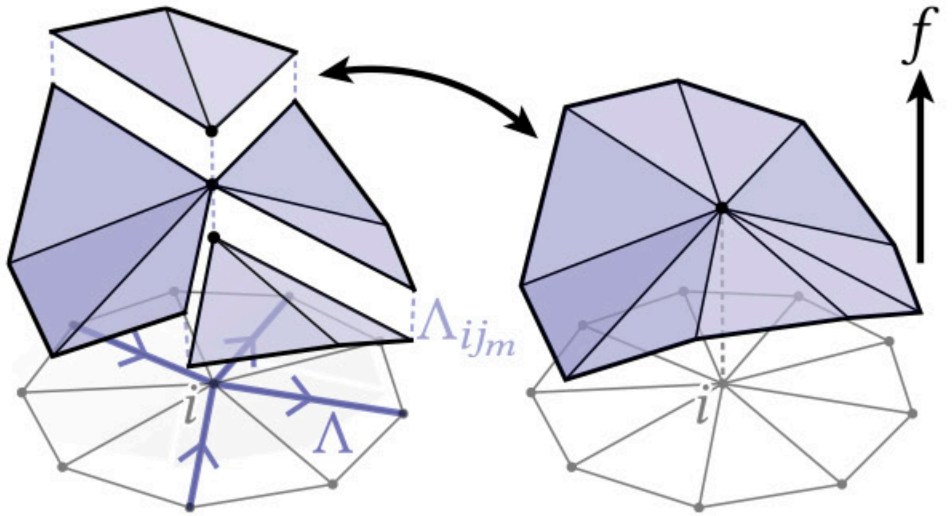


Value per corner →
one value per vertex!

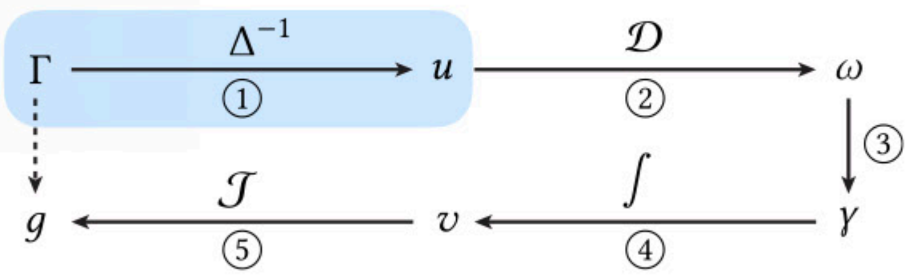
Apply change of variables and solve

$$Lf_0 = b$$

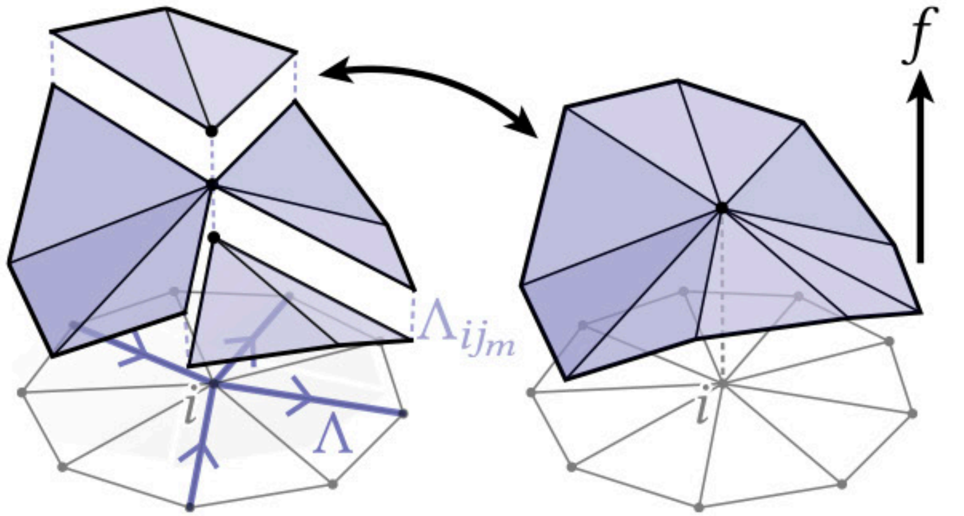
values per vertex

constant vector encoding per-corner jumps

Reduced coordinates



A jump harmonic function f is harmonic "up to jumps" Λ.



Value per corner →
one value per vertex!

Designing Quadrangulations with Discrete Harmonic Forms. Tong, Alliez, Cohen-Steiner, Desbrun (2006)

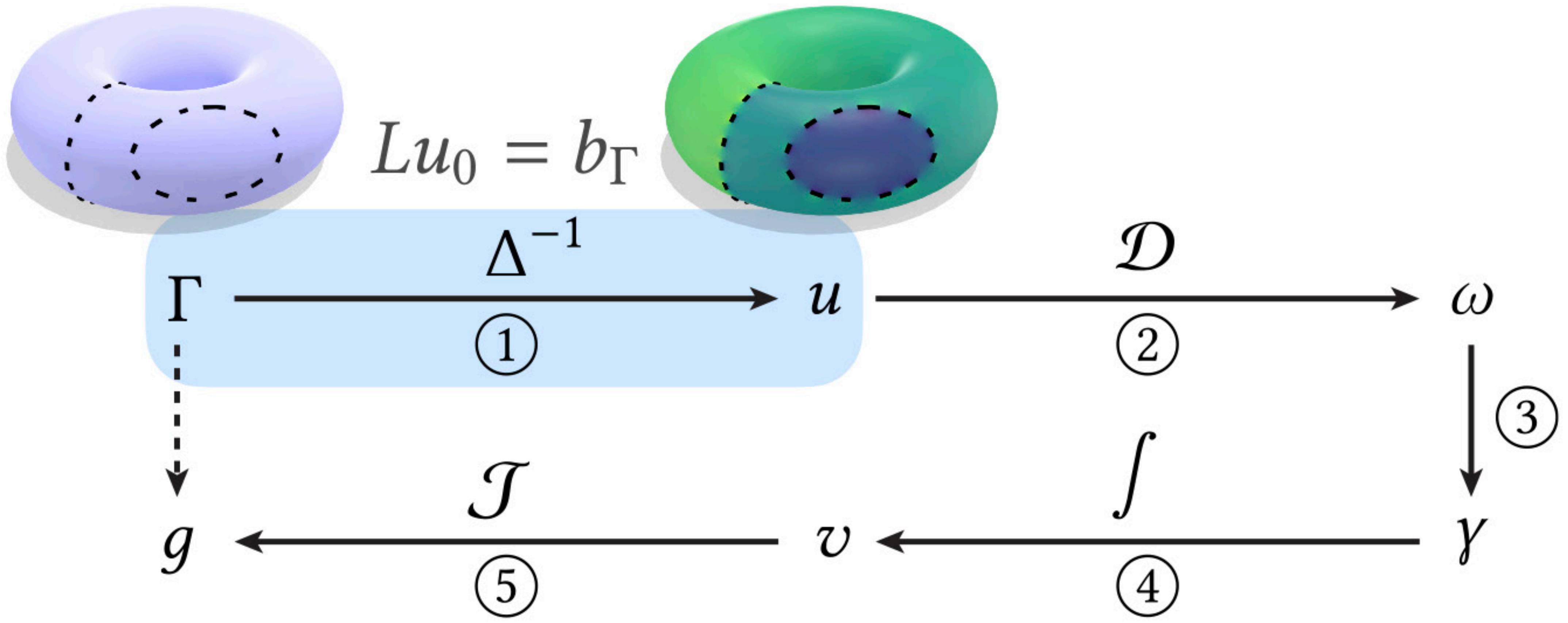
Apply change of variables and solve

$$L f_0 = b$$

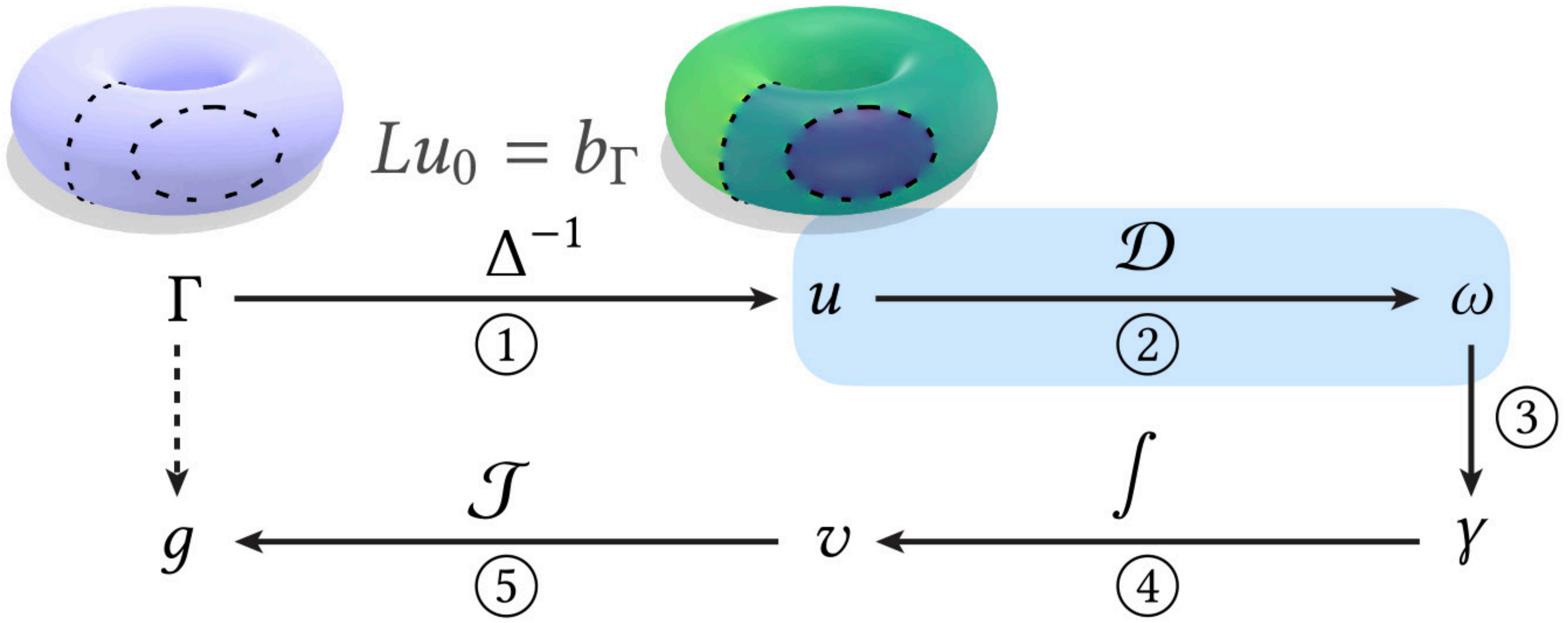
values per vertex

constant vector encoding per-corner jumps

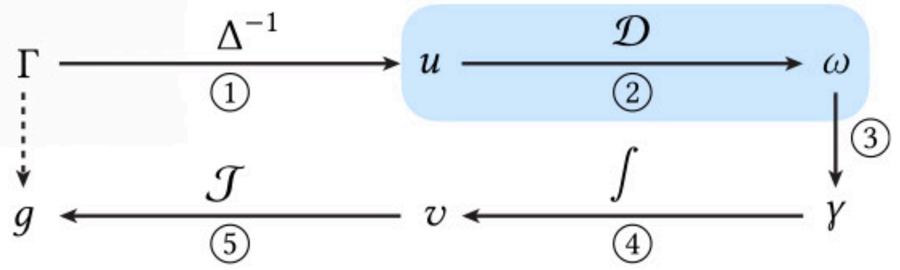
① Solve the jump Laplace equation for u

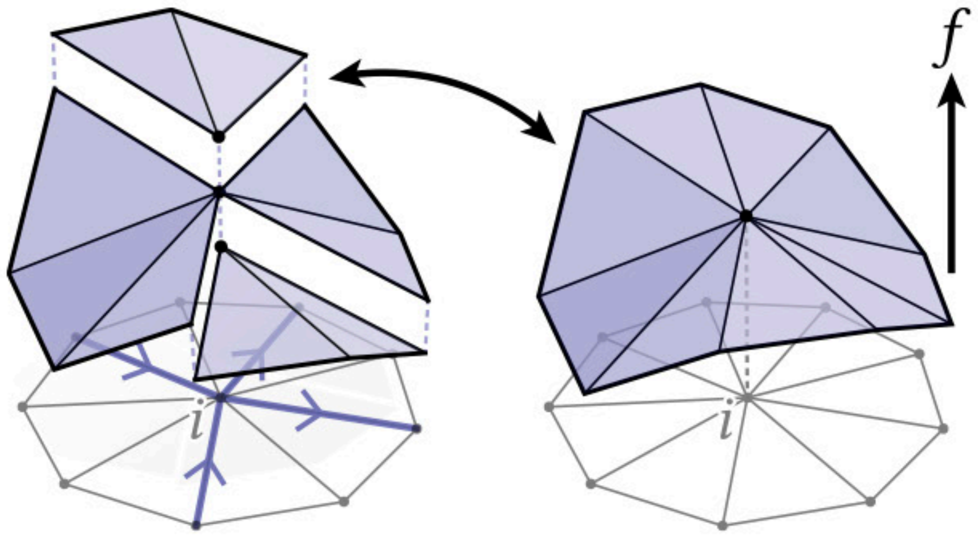


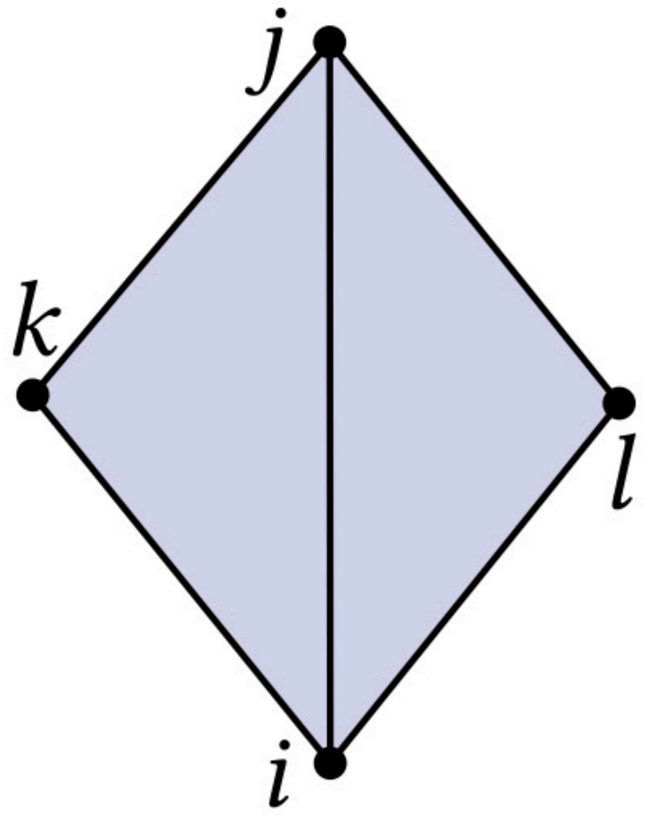
Jump harmonic functions → Derivatives...



Differentiation







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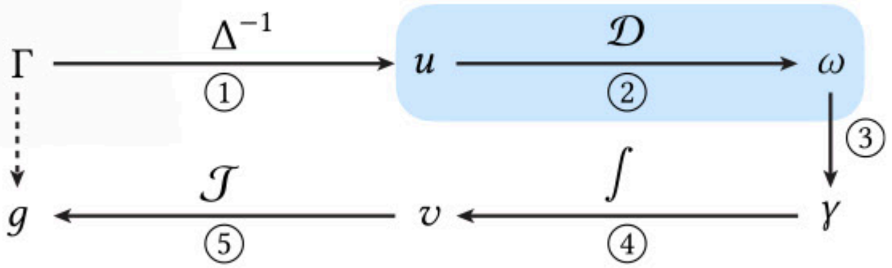
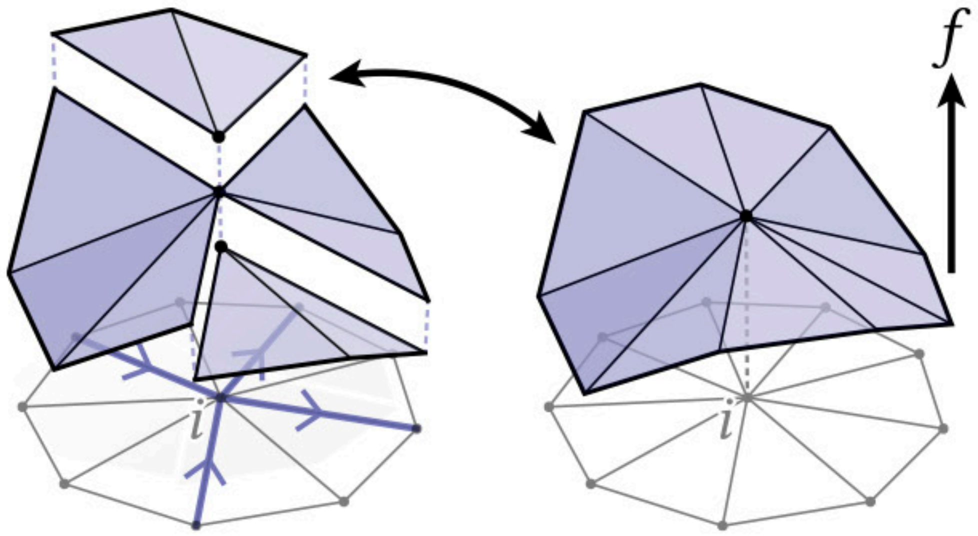
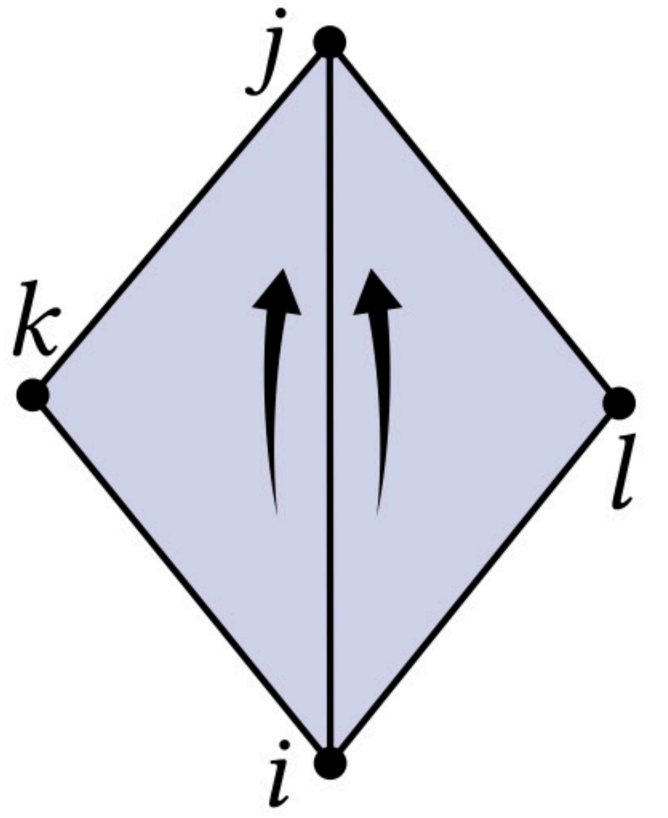
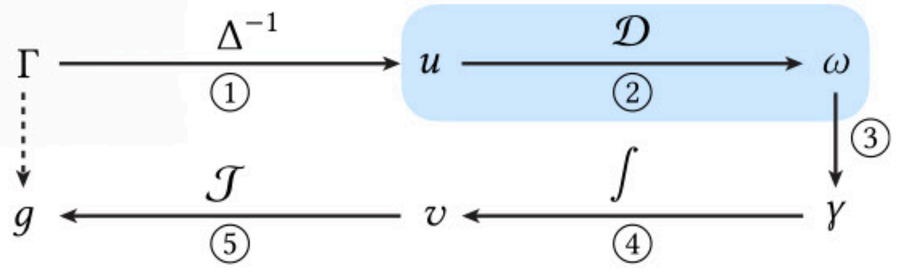
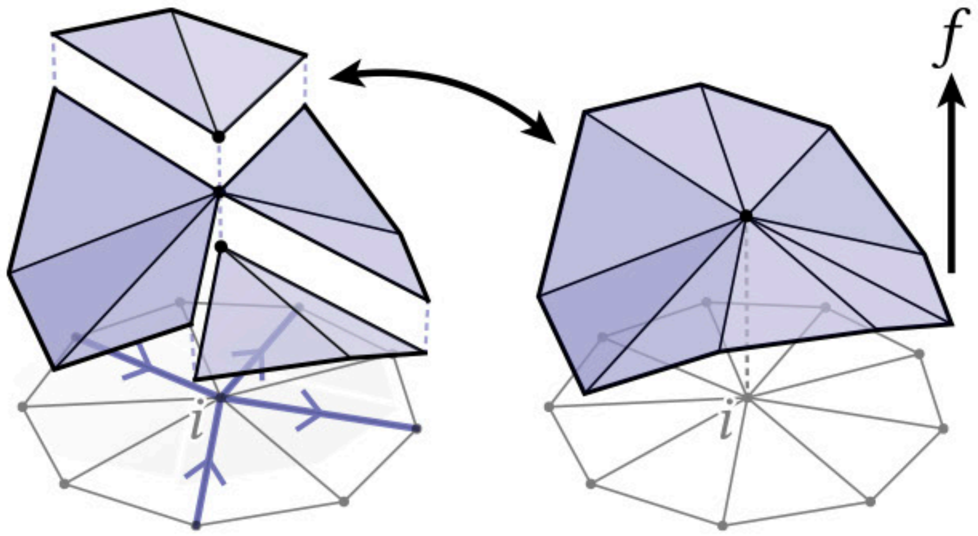


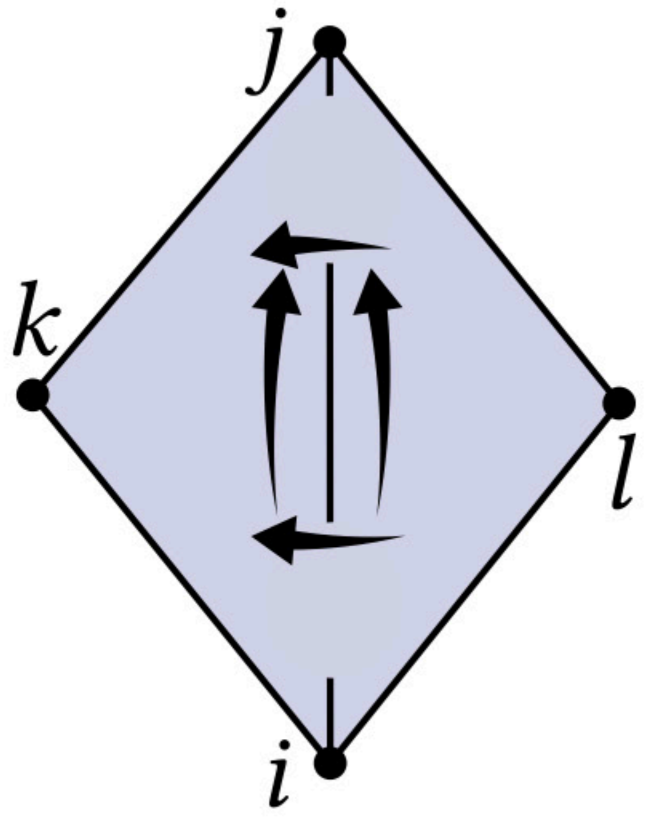




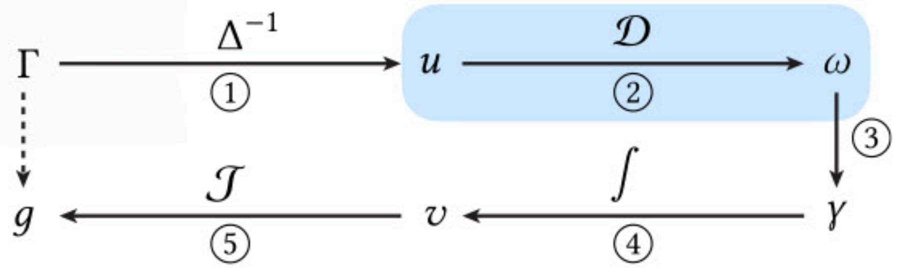
Differentiation

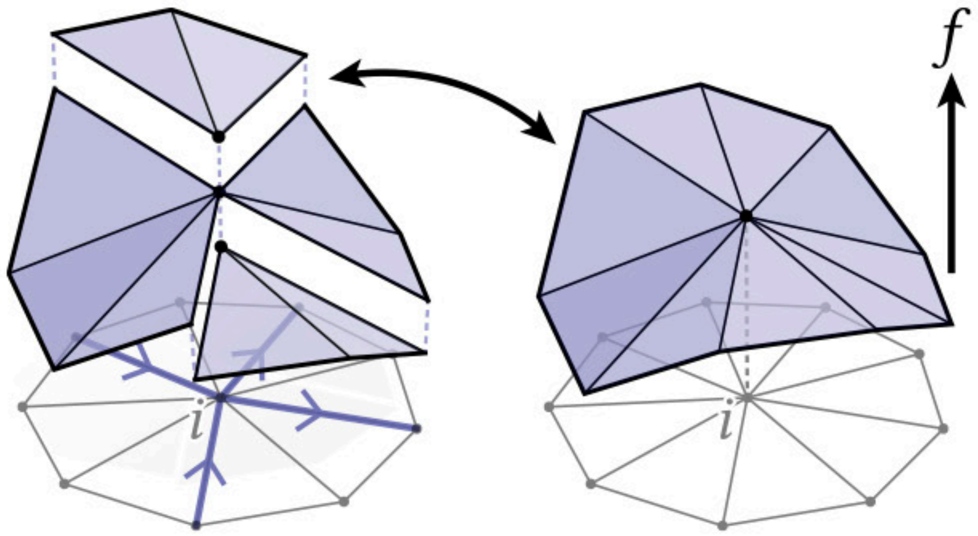






Differentiation

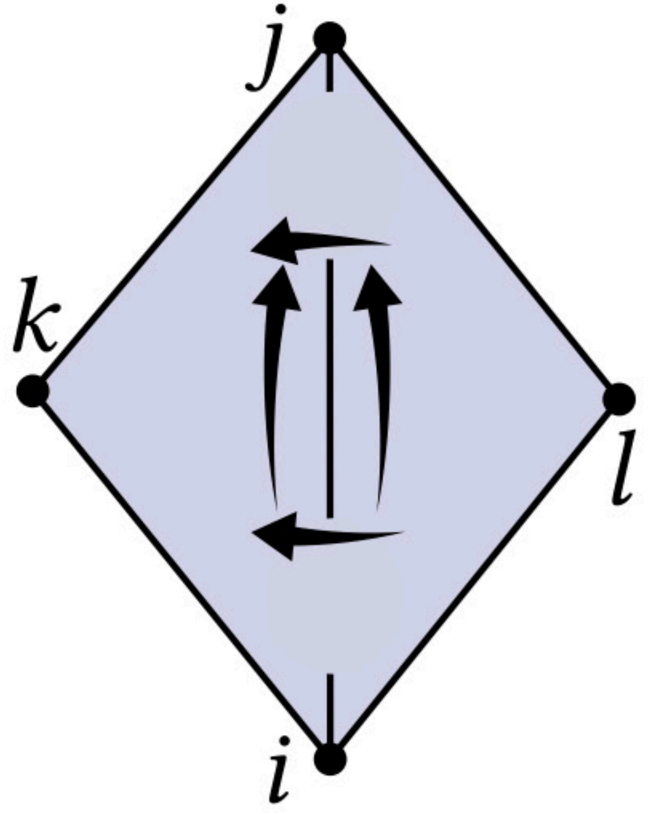




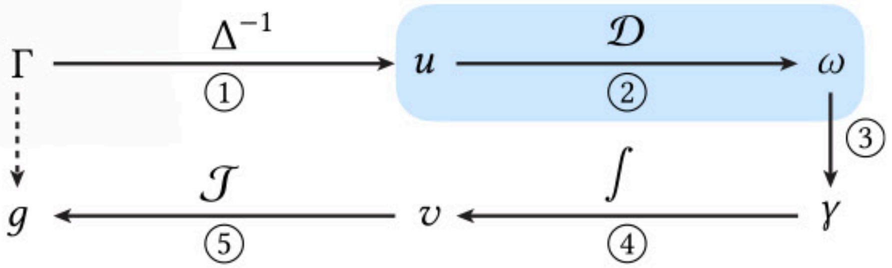
⇛

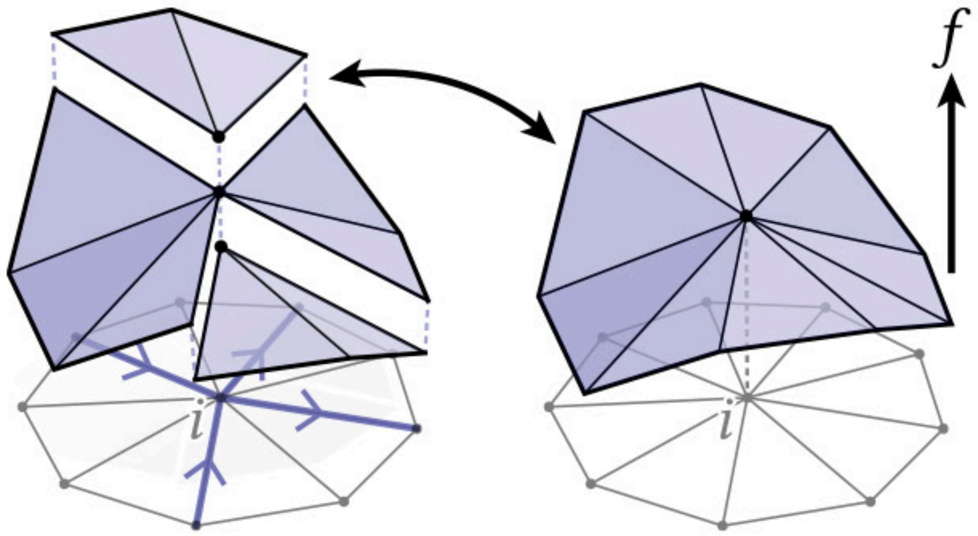
Jump compatibility condition:

$$f_i^{jk} - f_i^{lj} = f_j^{ki} - f_j^{il}$$



Differentiation

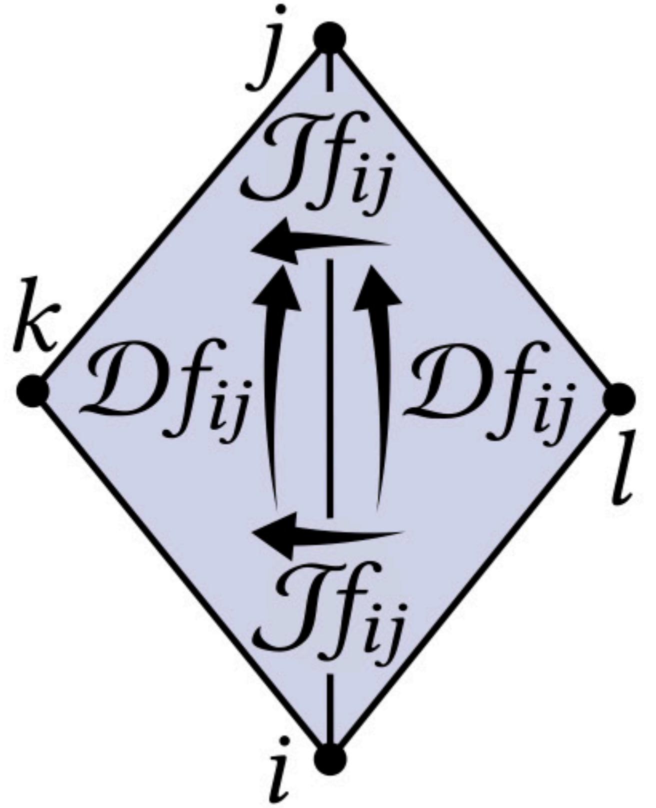




$\Rightarrow$

Jump compatibility condition:

$$f_i^{jk} - f_i^{lj} = f_j^{ki} - f_j^{il}$$

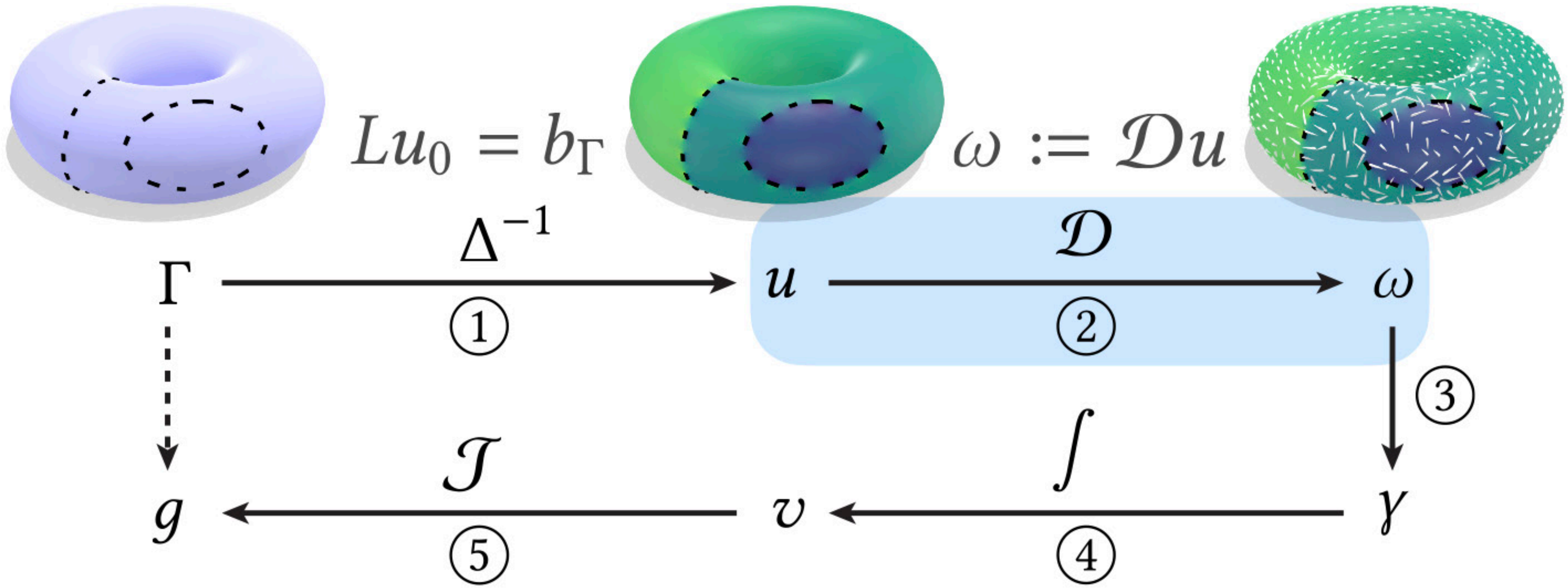


Darboux derivative: $(\mathcal{D}f)_{ij} := f_j^{ki} - f_i^{jk}$

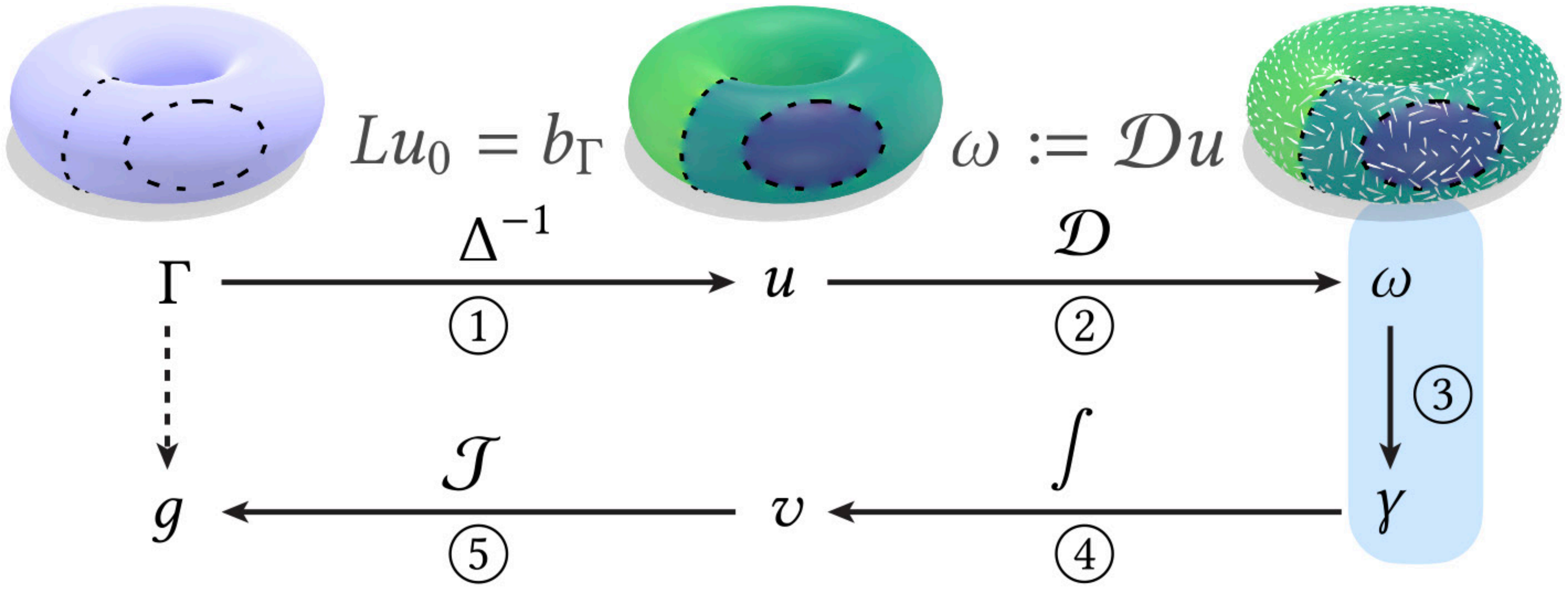
Jump derivative: $(\mathcal{J}f)_{ij} := f_i^{jk} - f_i^{lj}$

*subject to curve endpoints or nonmanifold edges

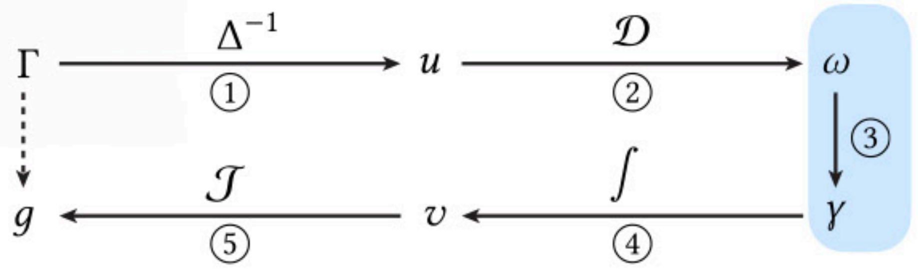
② Differentiate u



Derivative decomposition...

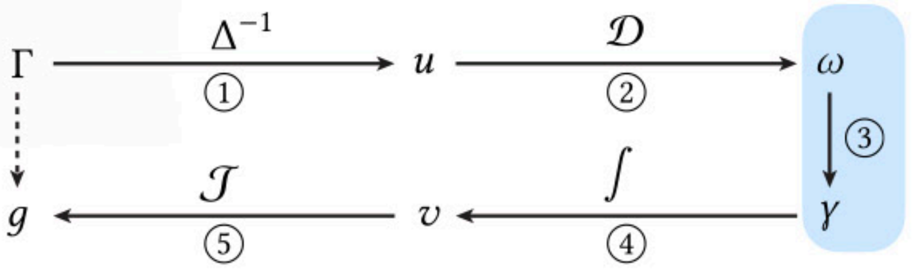


Hodge decomposition



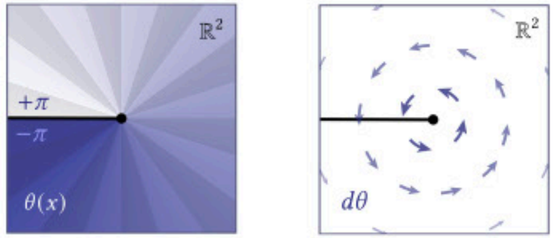
$$\omega = d\alpha + \delta\beta + \gamma$$

Hodge decomposition

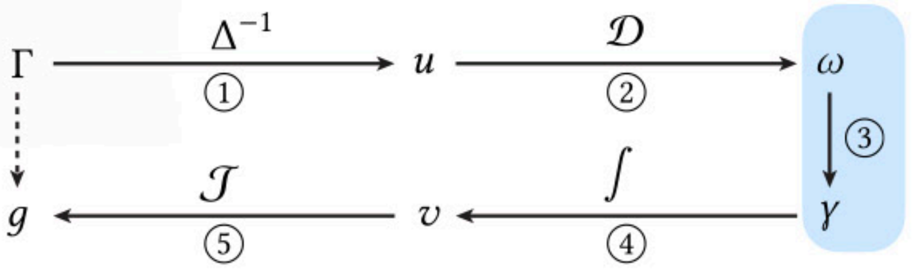


$$\omega = \cancelto{0}{d\alpha} + \delta\beta + \gamma$$

Lemma, Appendix A

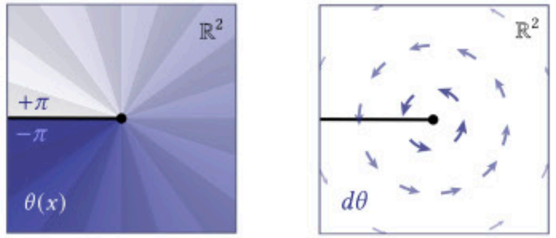


Hodge decomposition



$$\omega = \cancelto{0}{d\alpha} + \delta\beta + \gamma$$

Lemma, Appendix A

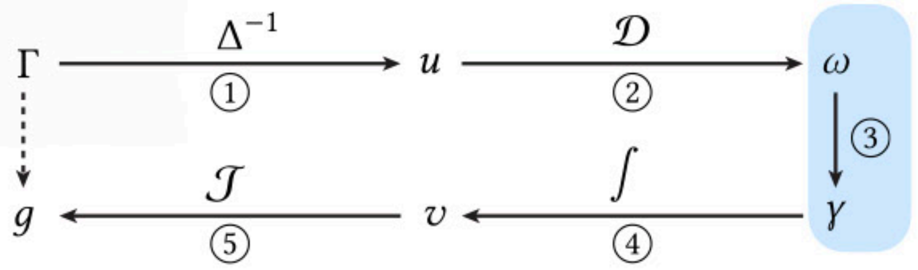


Solve Poisson equation: $\Delta_2 \beta = d_1 \omega$

$$\Delta_2 := d_1 *_1^{-1} d_1^T *_2$$

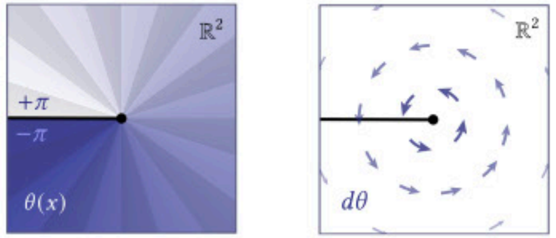
Discrete differential forms for computational modeling.
Desbrun, Kanso, Tong (2005)

Hodge decomposition



$$\omega = \cancel{d\alpha}^{\,0} + \delta\beta + \gamma$$

Lemma, Appendix A



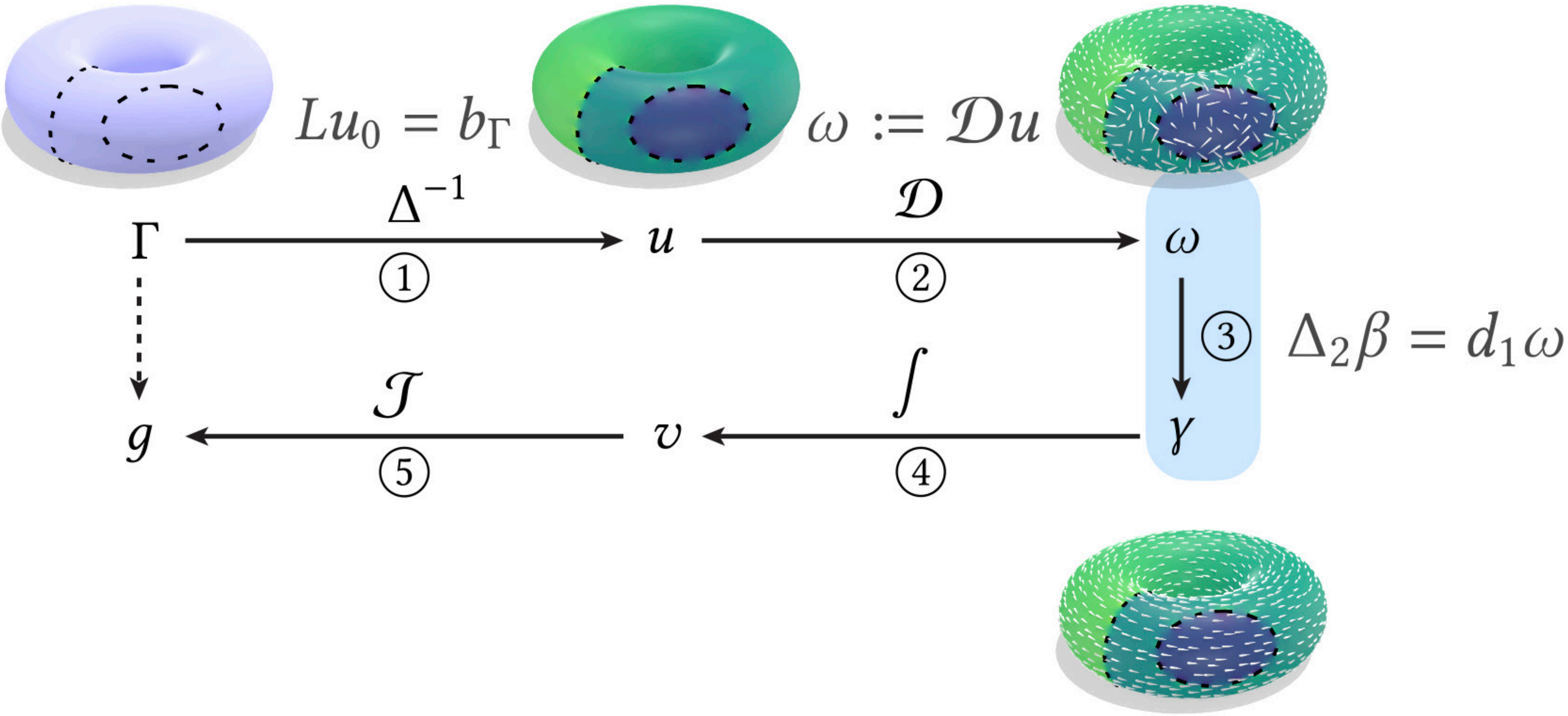
Solve Poisson equation: $\Delta_2 \beta = d_1 \omega$

$$\Delta_2 := d_1 *_1^{-1} d_1^T *_2$$

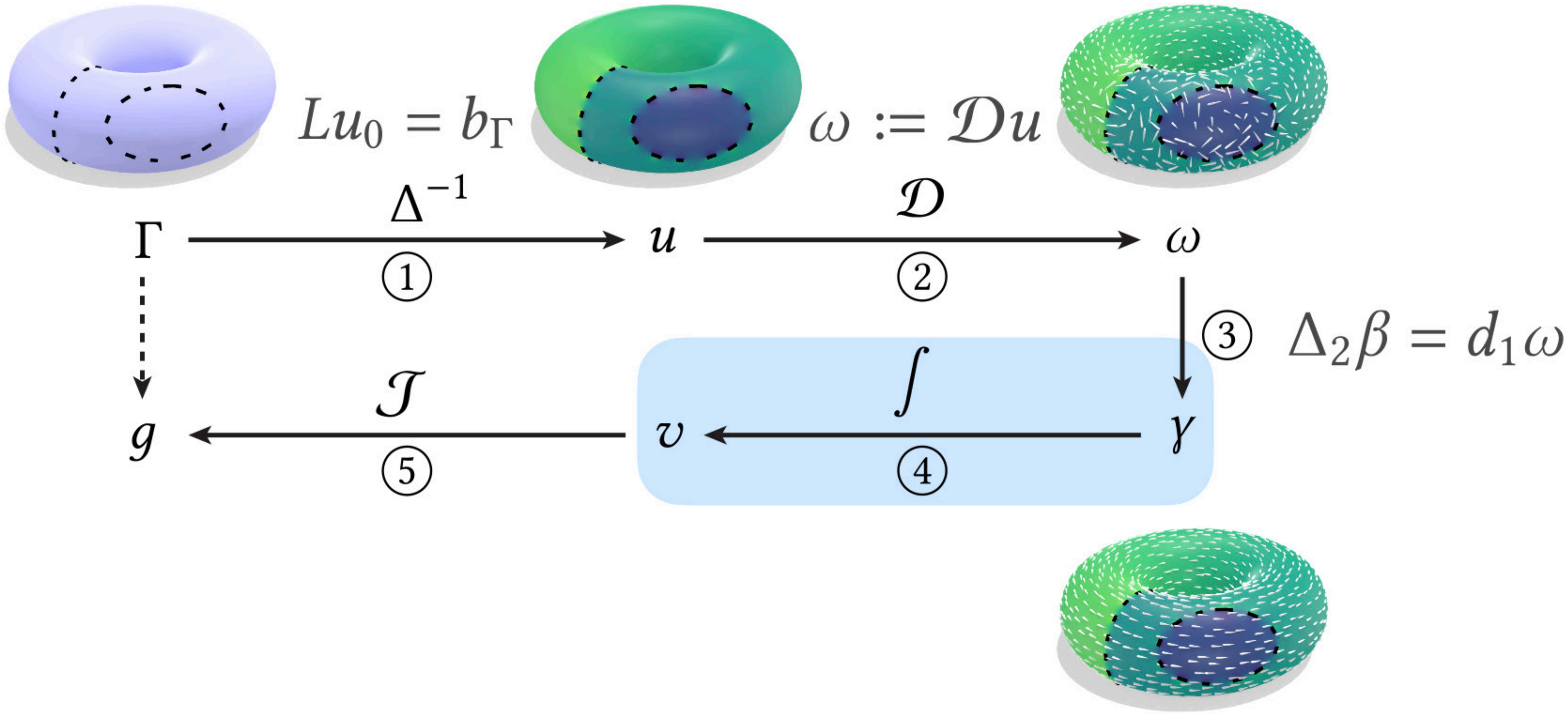
Get harmonic component: $\gamma \leftarrow \omega - \delta\beta$

Discrete differential forms for computational modeling.
Desbrun, Kanso, Tong (2005)

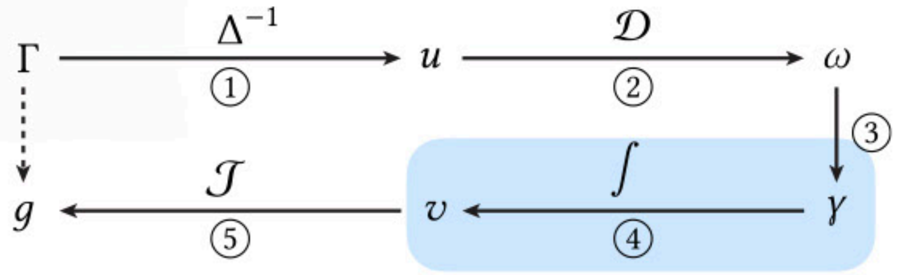
③ Hodge decompose ω



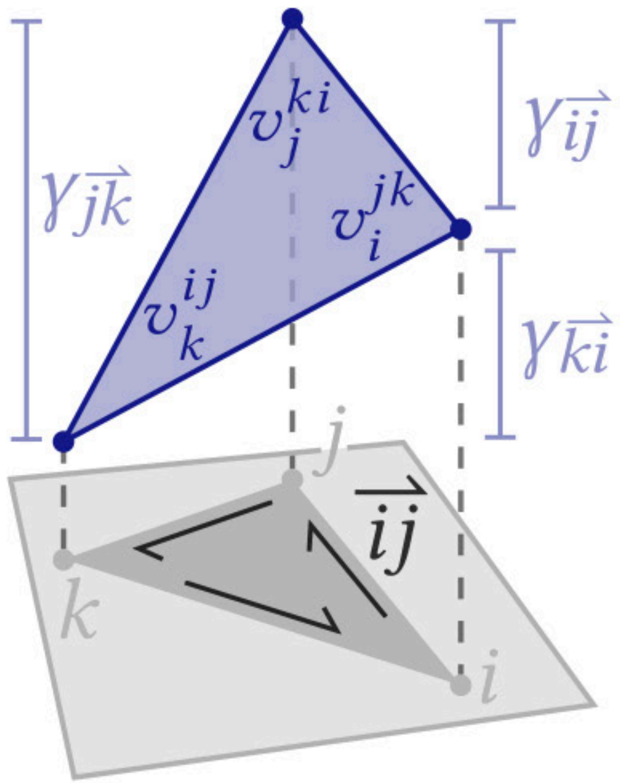
1-forms → Jump harmonic functions...



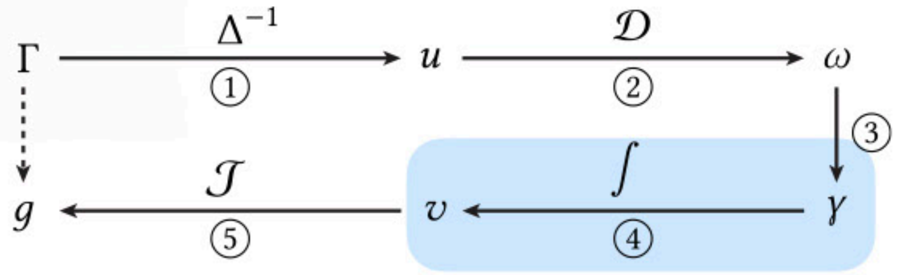
Integrating γ with jumps



First locally integrate γ in each triangle.



Integrating γ with jumps



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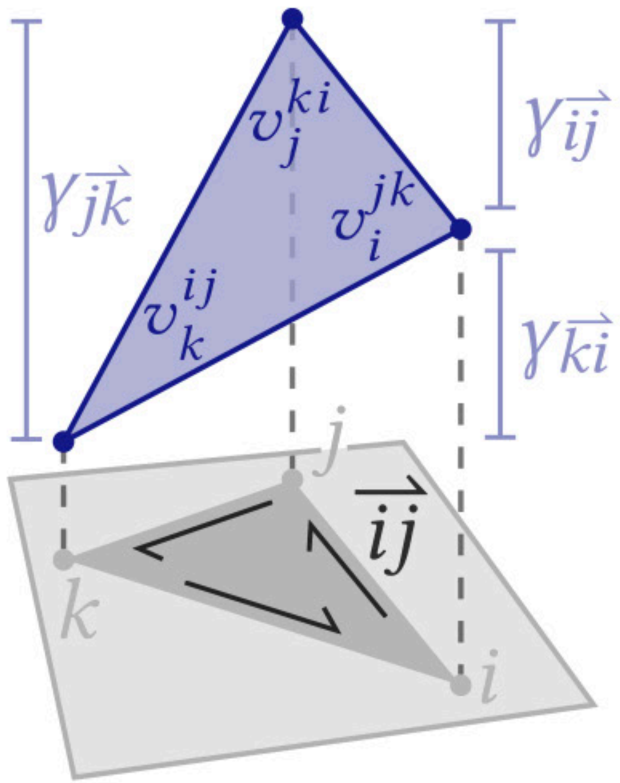
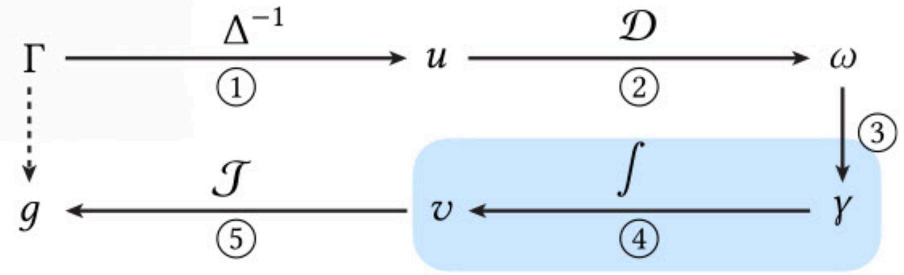
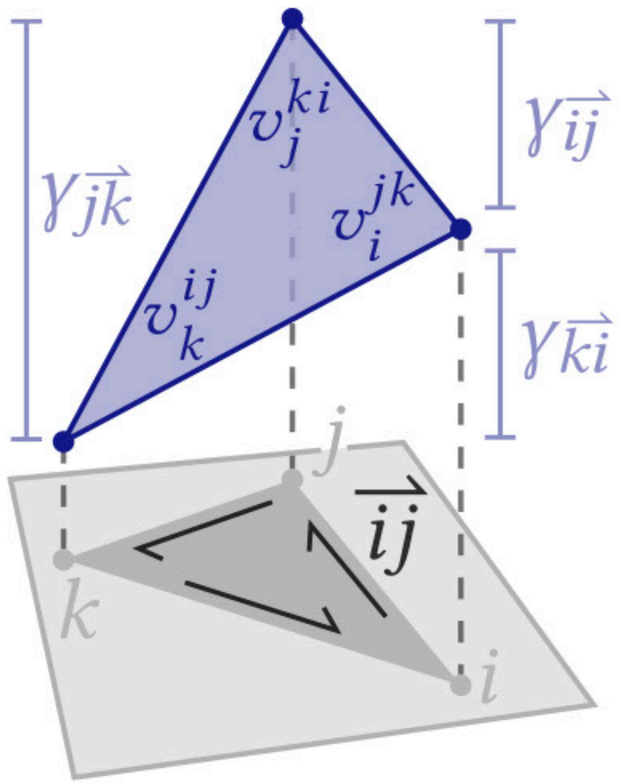


$$\mathring{v}_i^{jk} := 0$$

Integrating γ with jumps



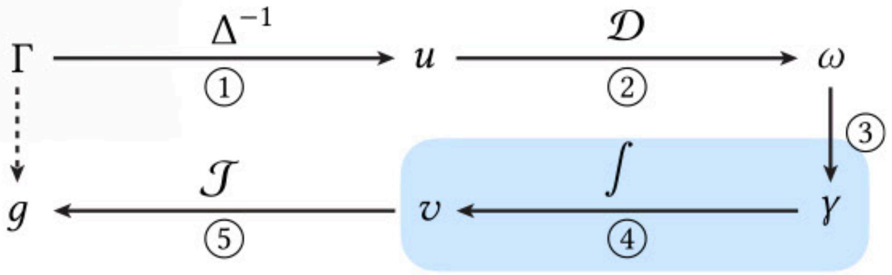
First locally integrate γ in each triangle.



$$\mathring{v}_i^{jk} := 0$$

$$\mathring{v}_j^{ki} := \gamma_{\overrightarrow{ij}}$$

Integrating γ with jumps



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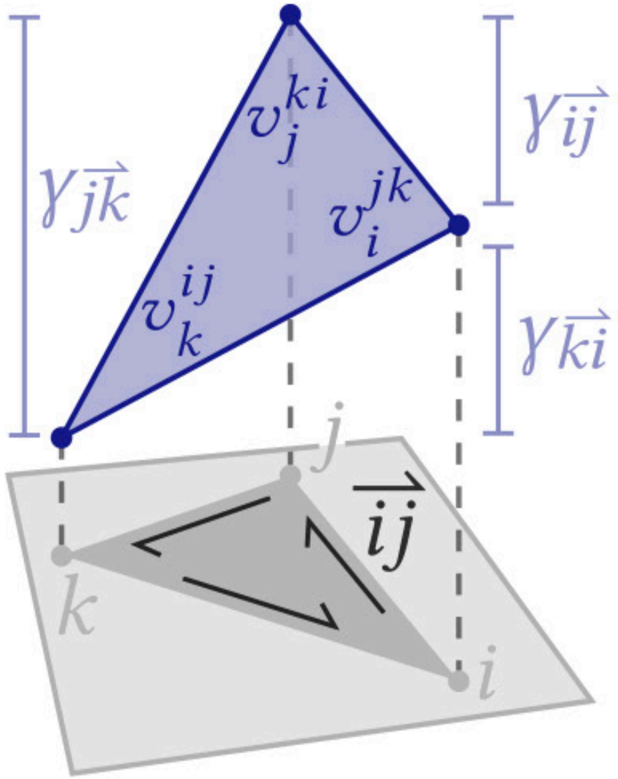
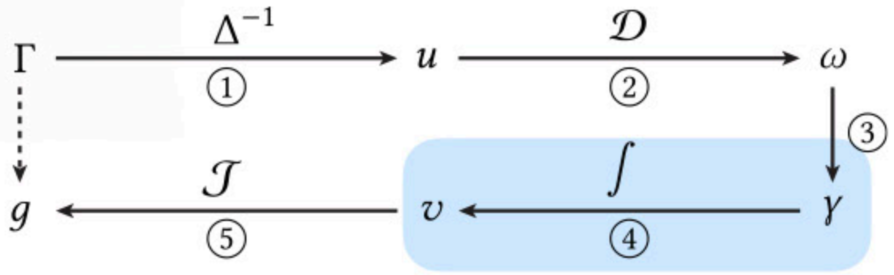


$$\mathring{v}_i^{jk} := 0$$

$$\mathring{v}_j^{ki} := \gamma_{\vec{ij}}$$

$$\mathring{v}_k^{ij} := \gamma_{\vec{ij}} + \gamma_{\vec{jk}}$$

Computing the residual function



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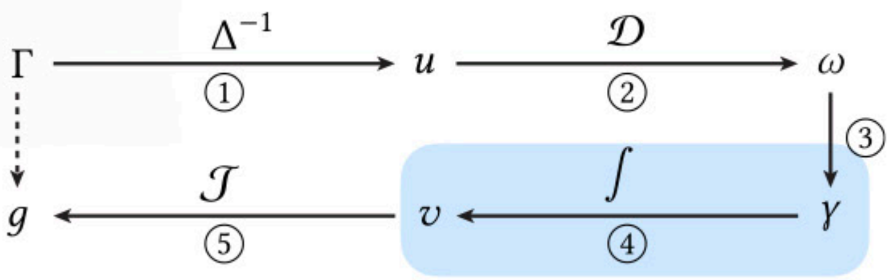


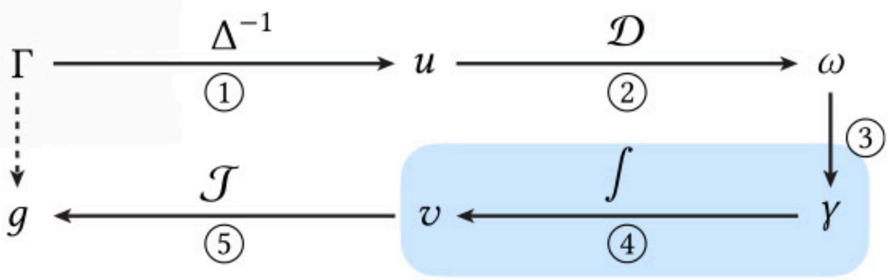
Applying a change of variable, optimize for $|F|$ per-face shifts:

$$\min_{\sigma \in \mathbb{R}^{|F|}} \sum_{ij \in \Gamma \cap E_{\text{int}}} \ell_{ij} |(\sigma_{ijk} - \sigma_{jil}) - s_{ij}| + \frac{1}{\varepsilon} \sum_{ij \in E_{\text{int}} \setminus \Gamma} \ell_{ij} |(\sigma_{ijk} - \sigma_{jil}) - s_{ij}|$$

$$\text{s.t.} \quad 0 \leq \frac{(\sigma_{ijk} - \sigma_{jil}) - s_{ij}}{\Gamma_{ij}} \leq 1, \quad \forall ij \in \Gamma.$$

linear program

Computing the residual function



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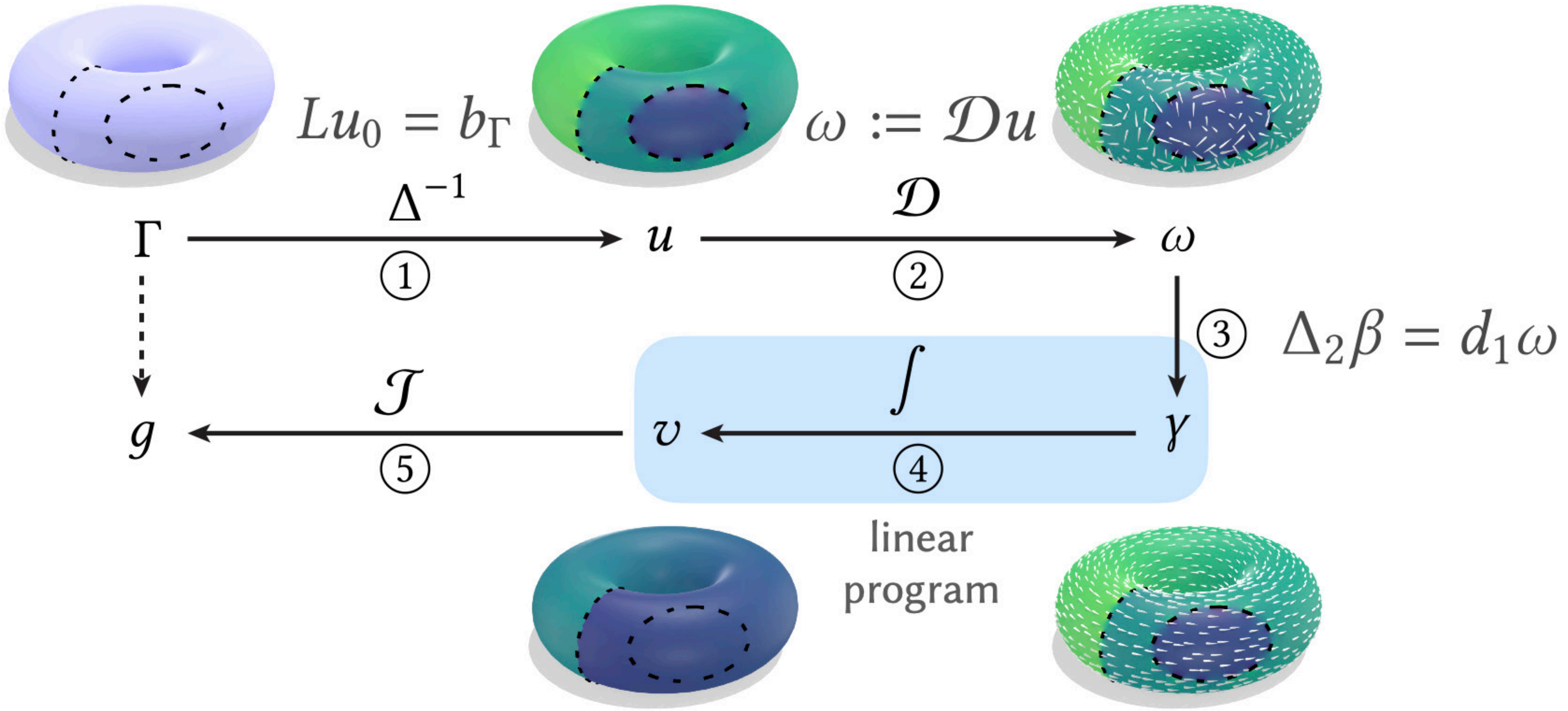
$$\text{s.t.} \quad 0 \leq \frac{(\sigma_{ijk} - \sigma_{jil}) - s_{ij}}{\Gamma_{ij}} \leq 1, \quad \forall ij \in \Gamma.$$

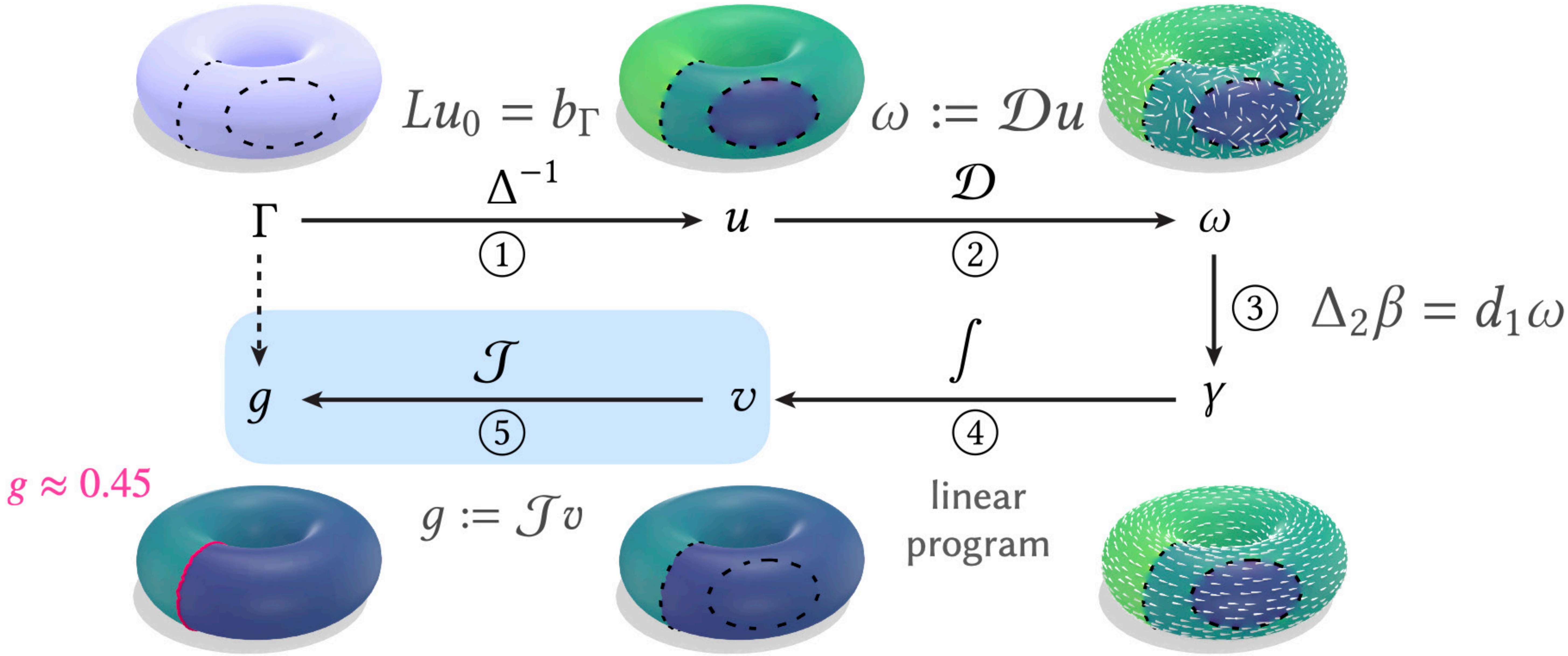
linear program

Afterwards, recover solution v.

④ Compute the residual function



⑤ Decompose curve



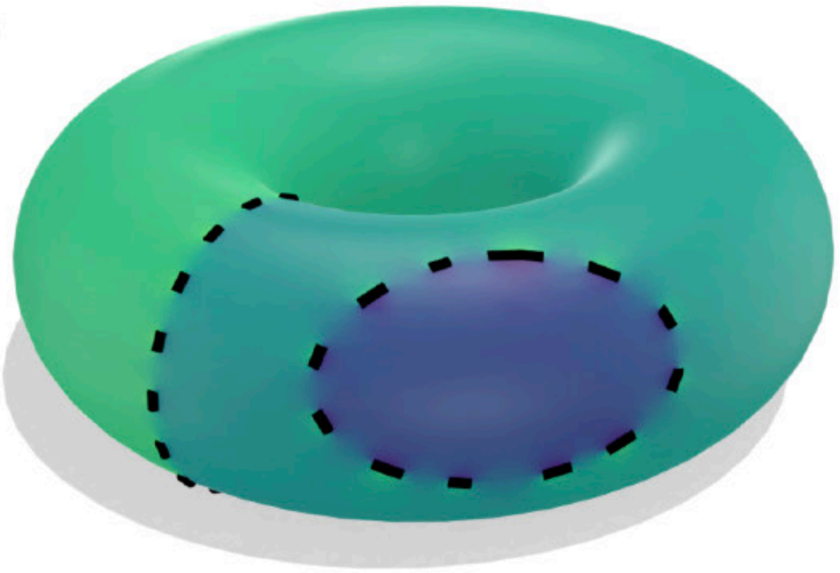
Winding number function

Solve jump Laplace equation with jumps $\Gamma - \mathcal{J}v$ to obtain w.

Winding number function

Solve jump Laplace equation with jumps $\Gamma - \mathcal{J}v$ to obtain w.

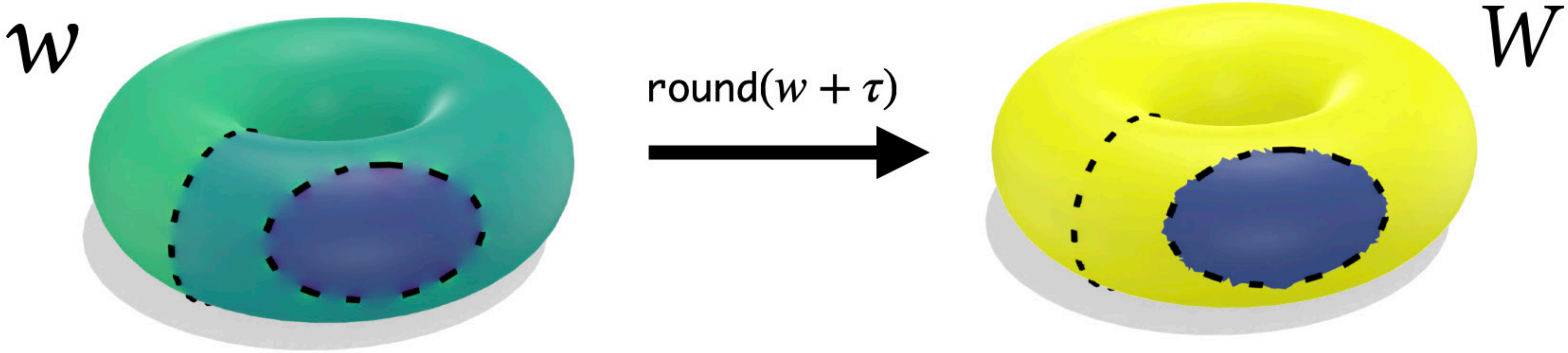
w



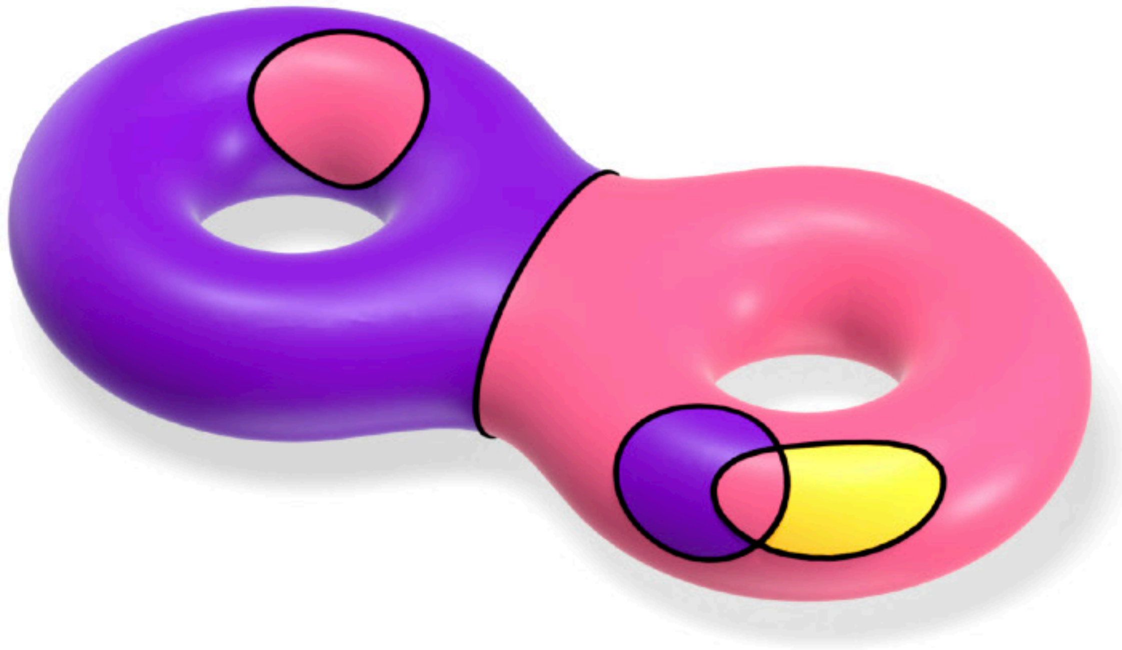
Winding number function

Solve jump Laplace equation with jumps $\Gamma - \mathcal{J} v$ to obtain w.

We compute a global shift τ such that $w + \tau$ is integer along Γ.

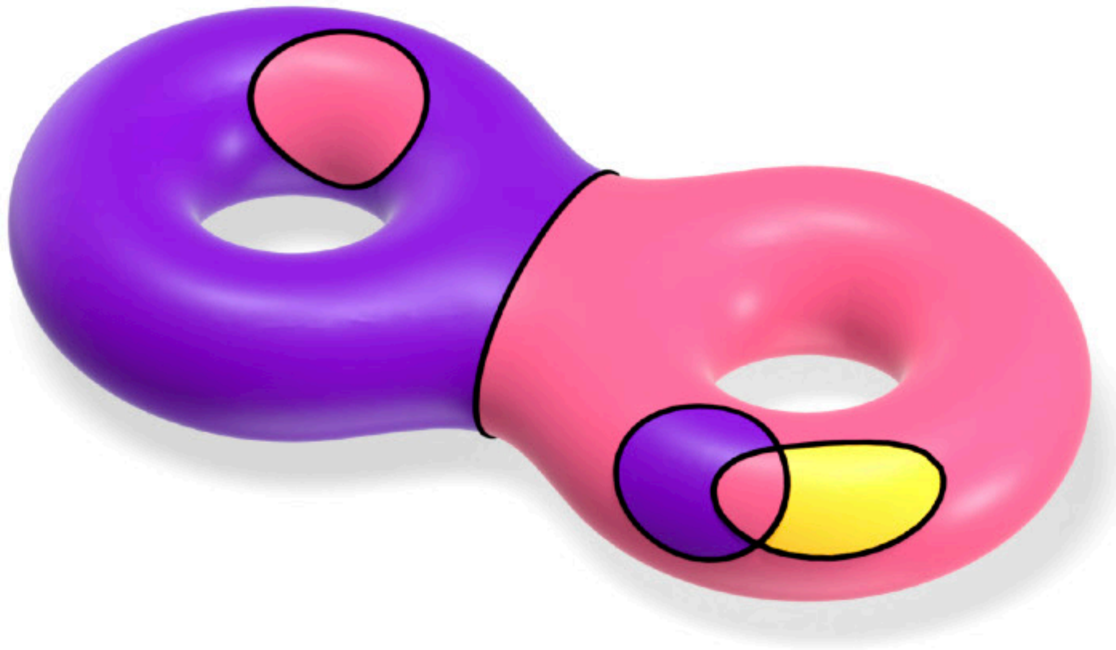


Summary



Summary

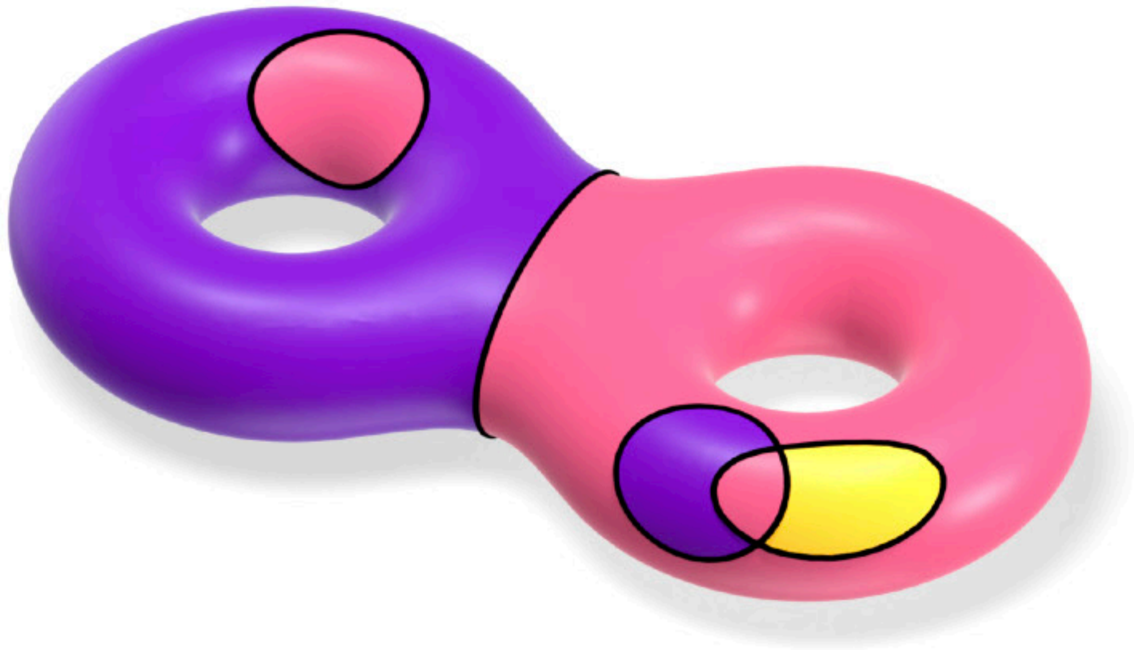
It's difficult to reason about the homology class of *broken* curves.



Summary

It's difficult to reason about the homology class of *broken* curves.

Instead of processing curves directly, we process functions *dual* to curves using *de Rham cohomology.*



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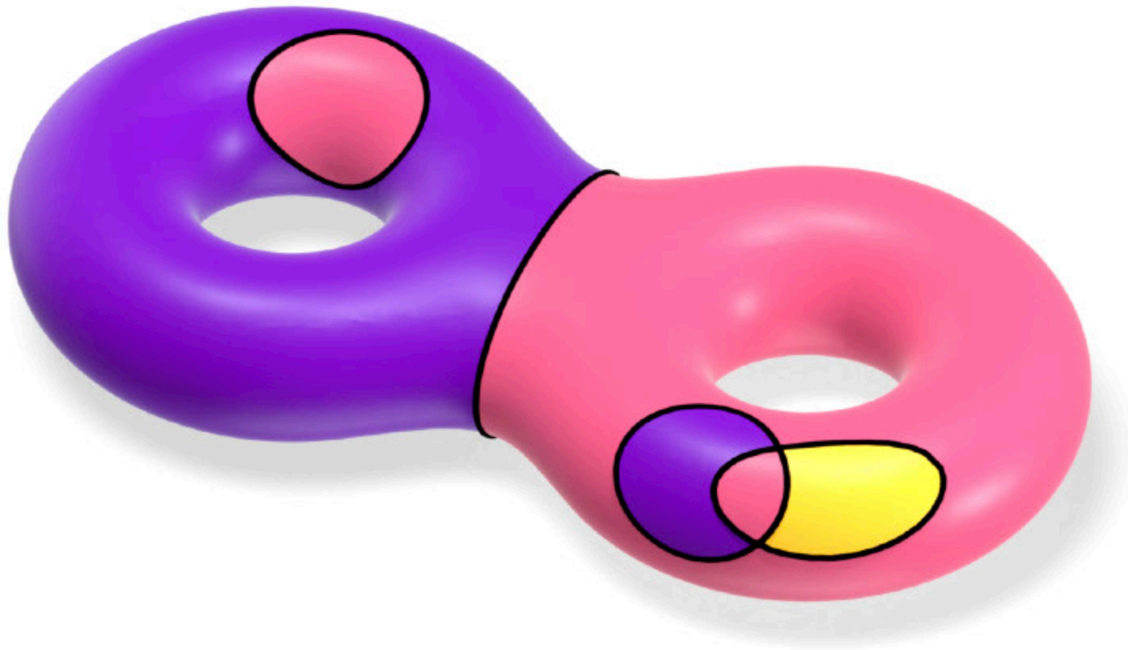

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We map from functions back to curves, yielding integer region labels & identification of nonbounding components.

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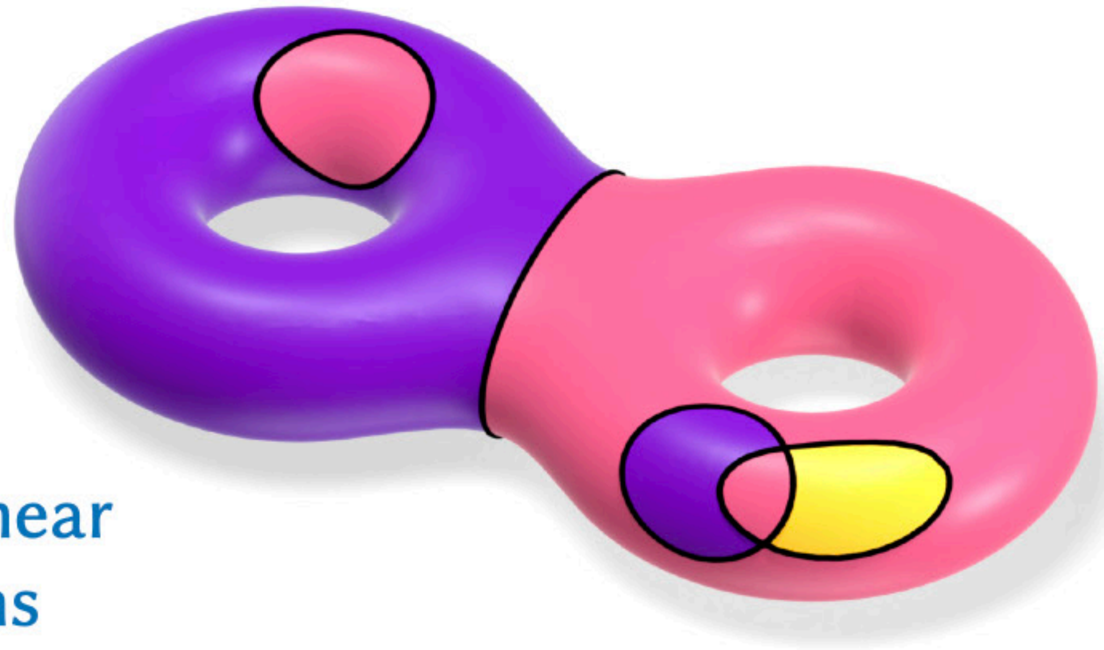
sparse linear systems



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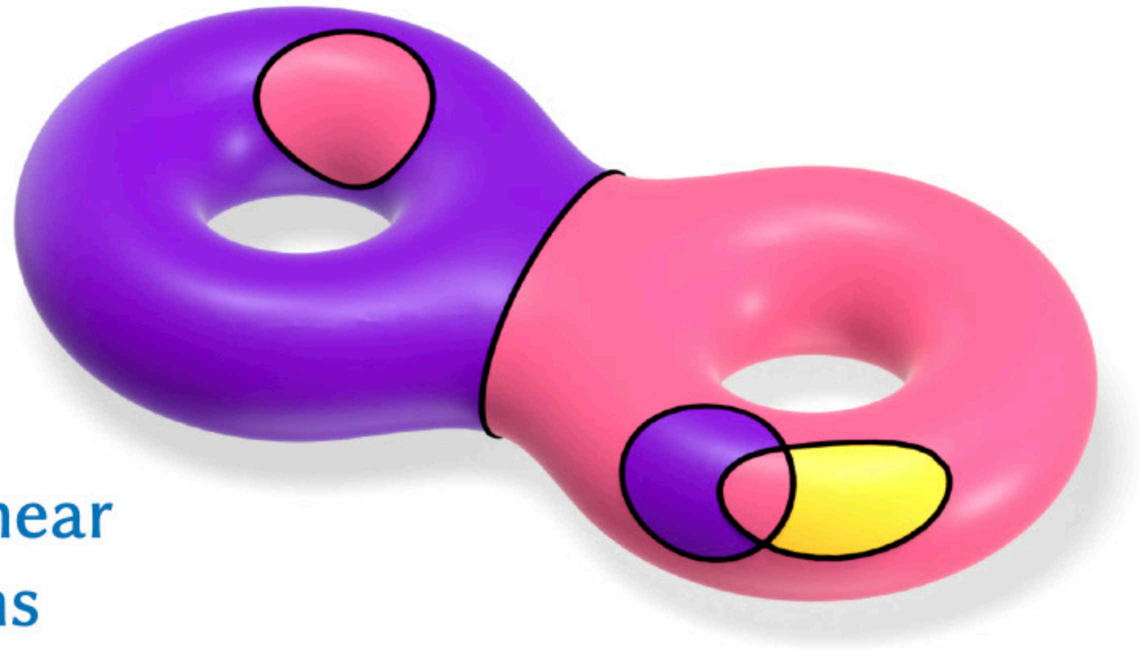
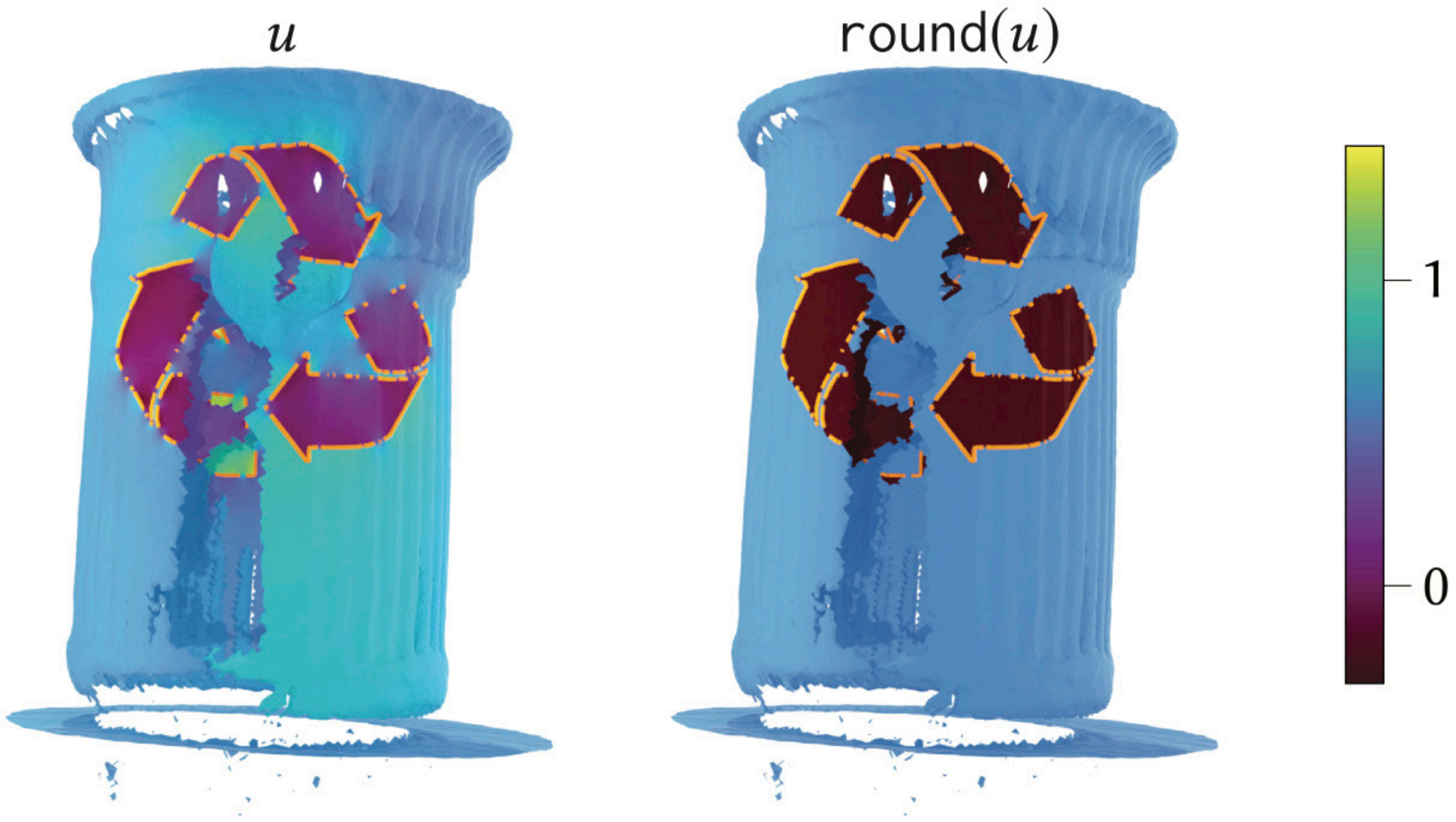


We map from functions back to curves, yielding integer region labels & identification of nonbounding components.

sparse linear program

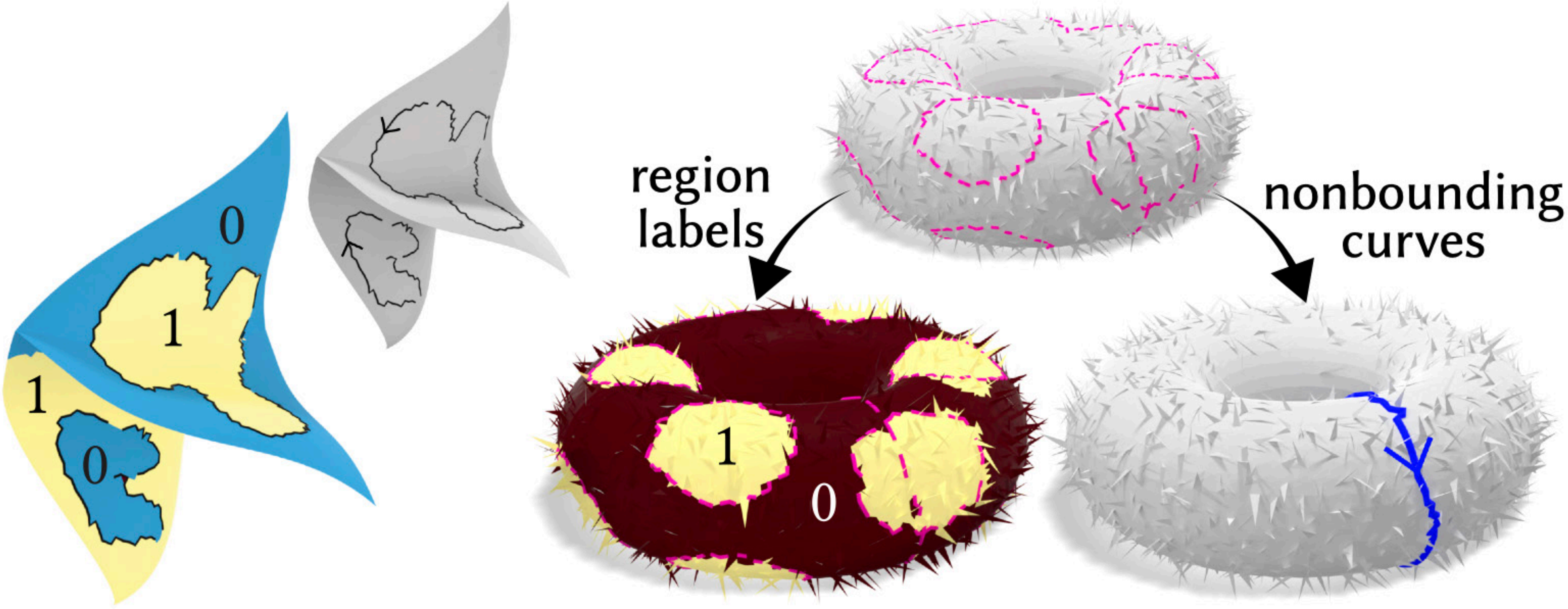
RESULTS

Robustness to defects in both Γ and M



holes, scanner noise

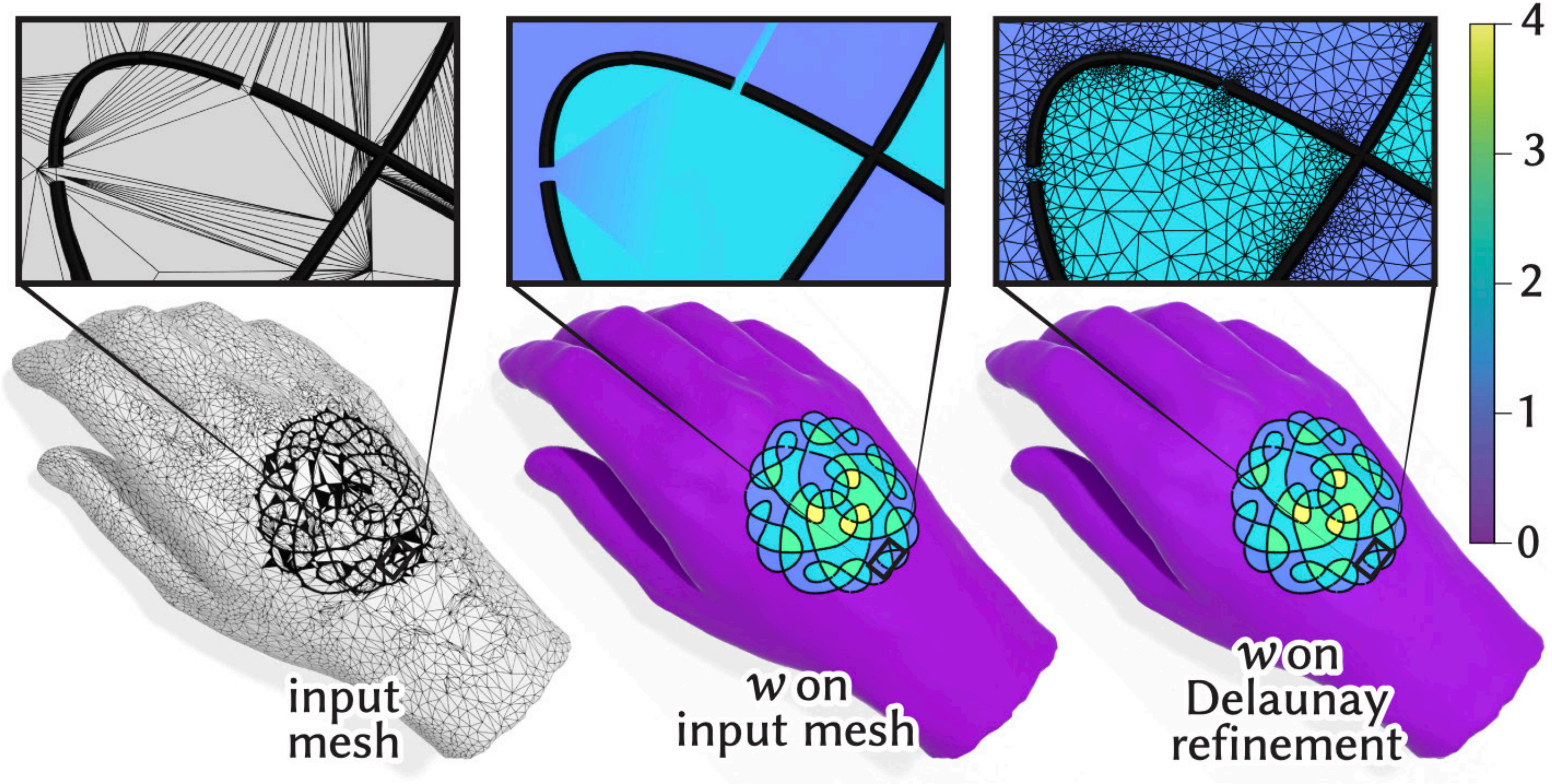
Robustness to defects in both Γ and M



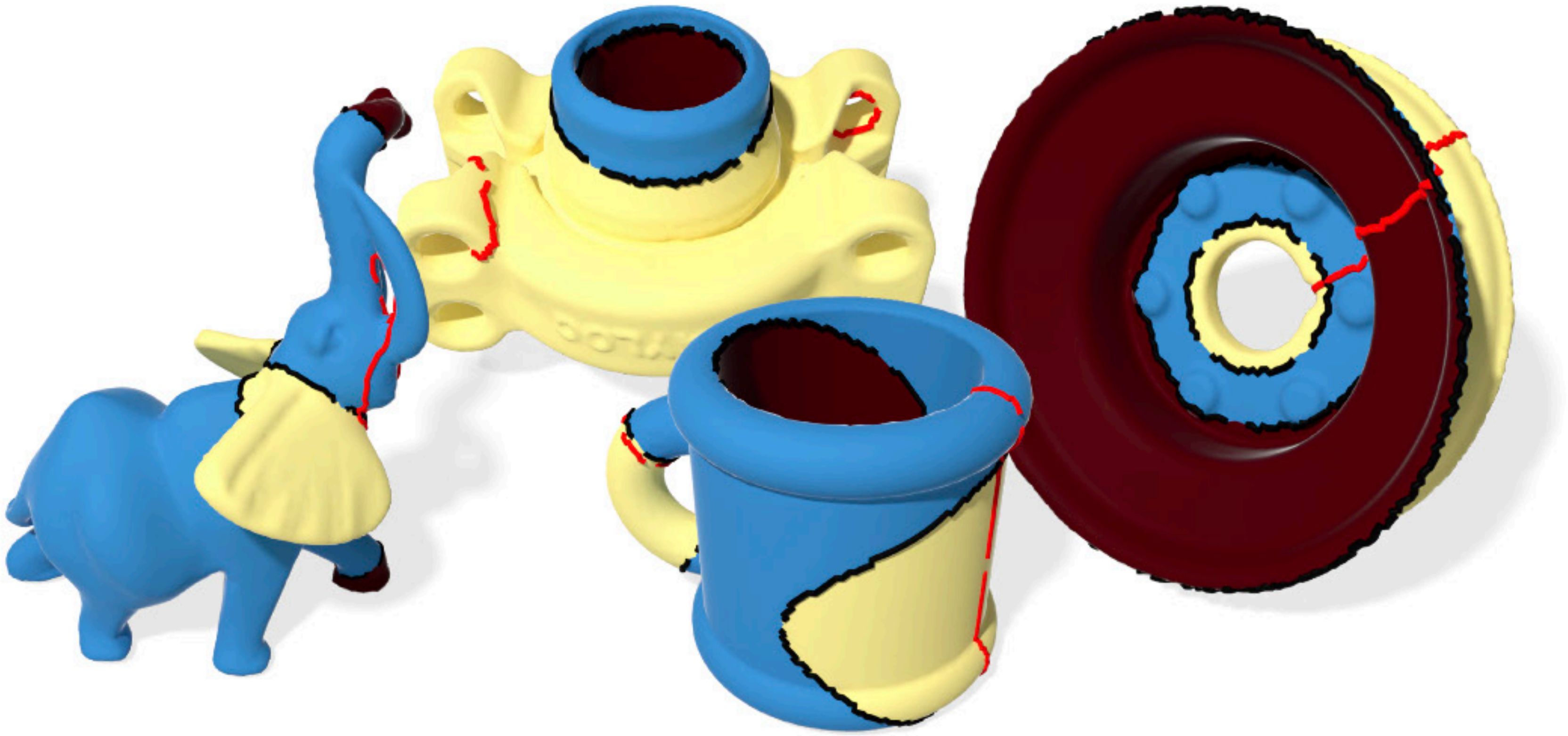
highly non-manifold surfaces

Intrinsic retriangulation

Navigating Intrinsic Triangulations. Sharp, Soliman, Crane (2019)
Integer Coordinates for Intrinsic Geometry Processing. Gillespie, Sharp, Crane (2021)

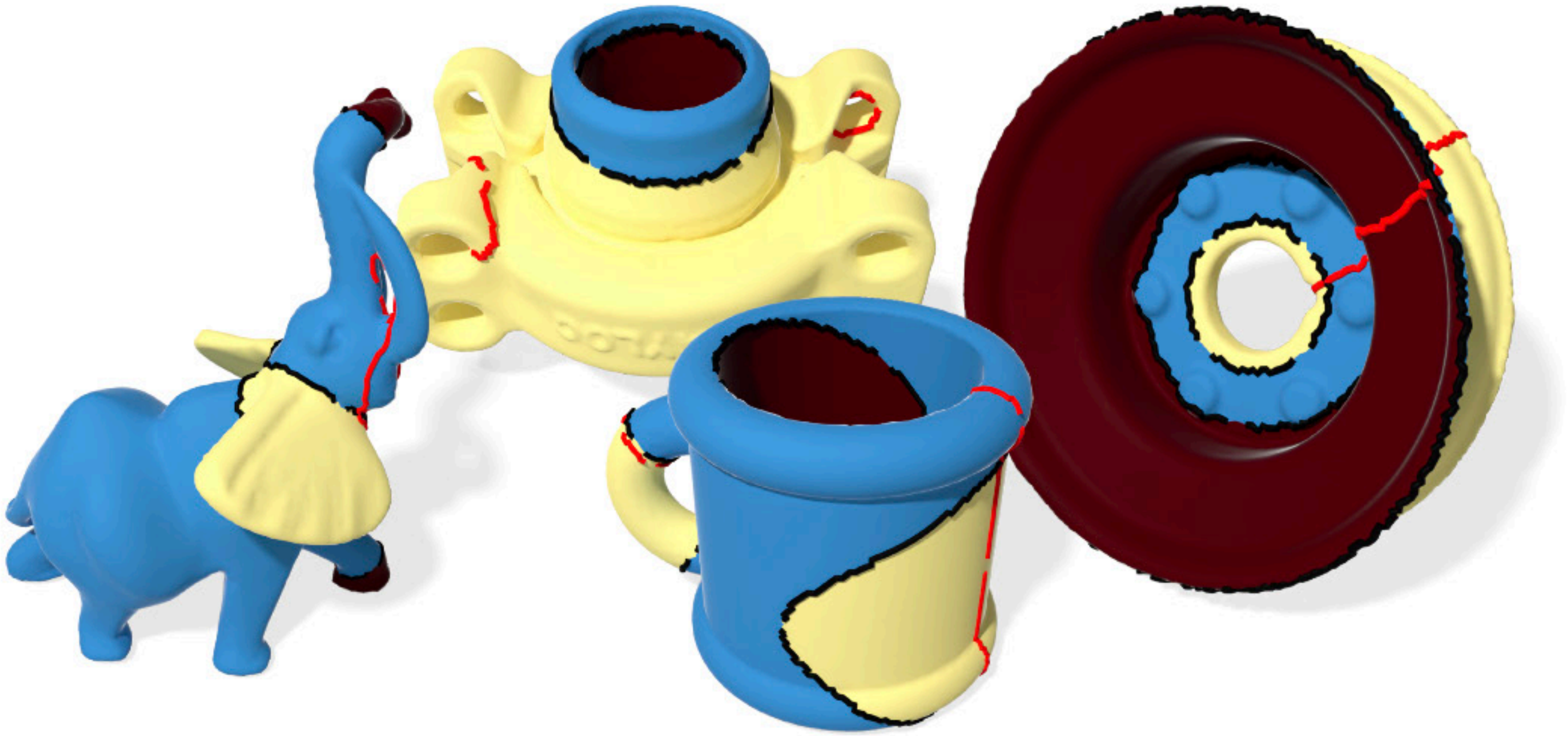


Evaluation



Benchmark setup

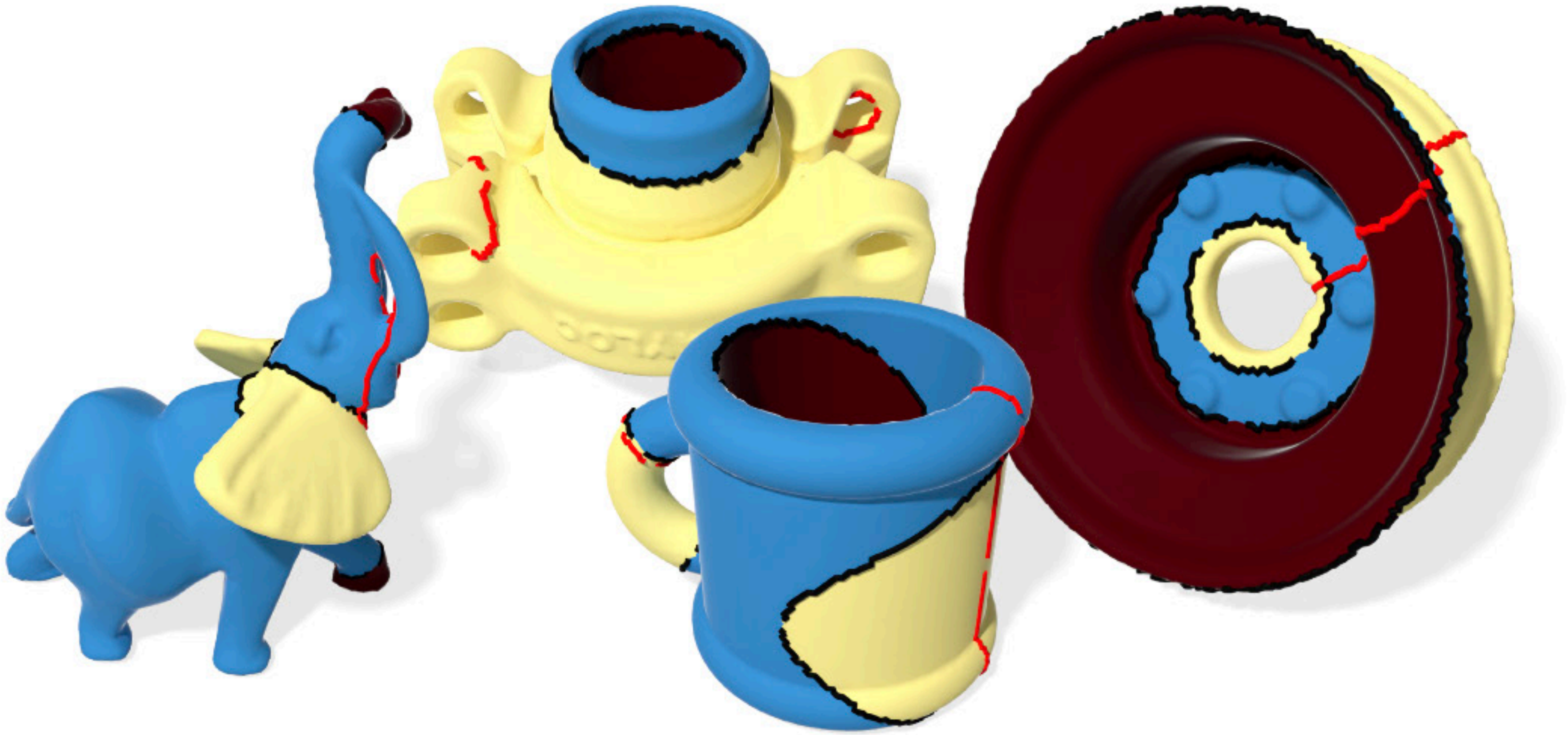
Evaluation



Benchmark setup

(1) Generate random ground-truth regions, take boundaries *[sub-levelsets of low-frequency Laplacian eigenfunctions]*

Evaluation



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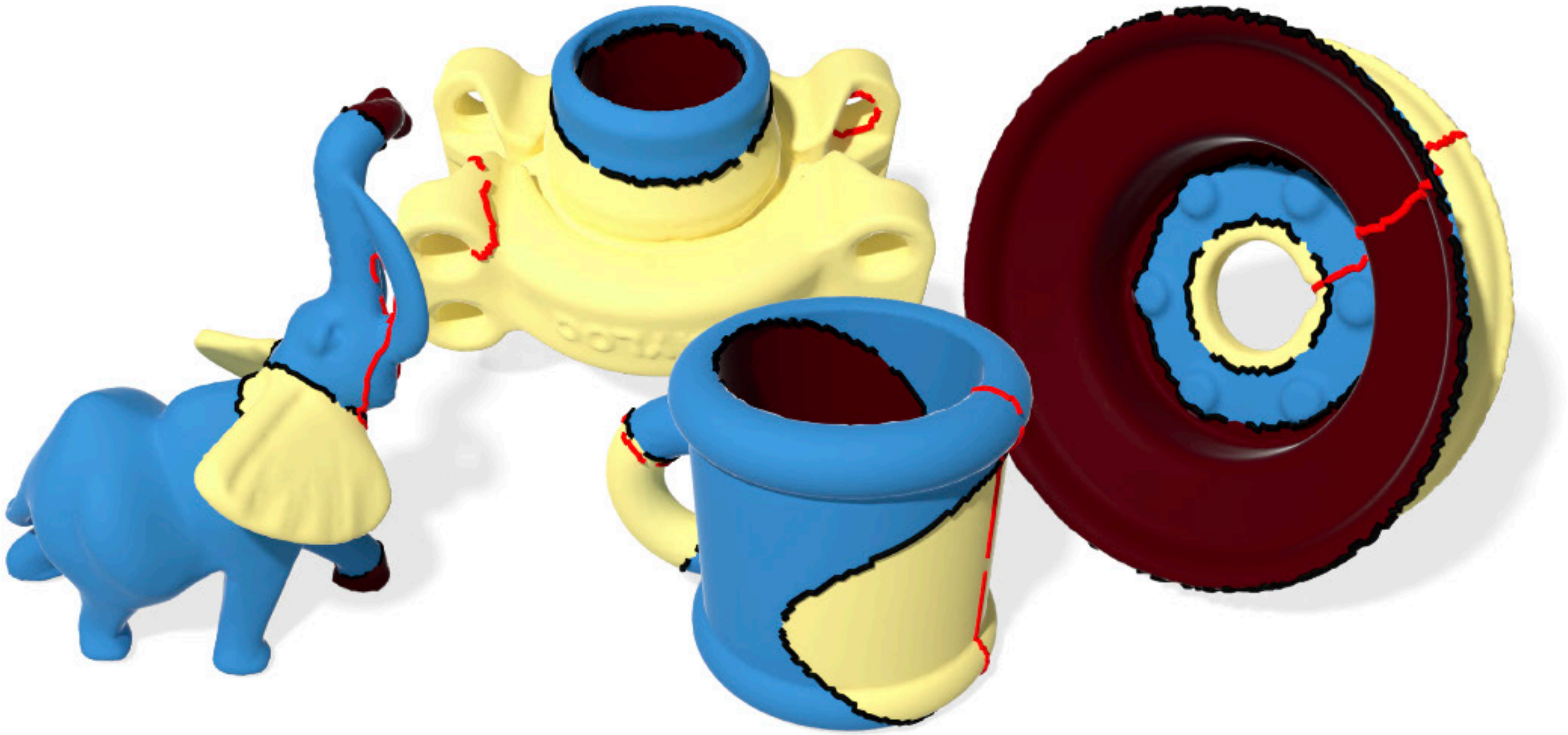
(1) Generate random ground-truth regions, take boundaries *[sub-levelsets of low-frequency Laplacian eigenfunctions]*

(2) Add nonbounding loops *[compute homology basis, select random subset]*

Evaluation



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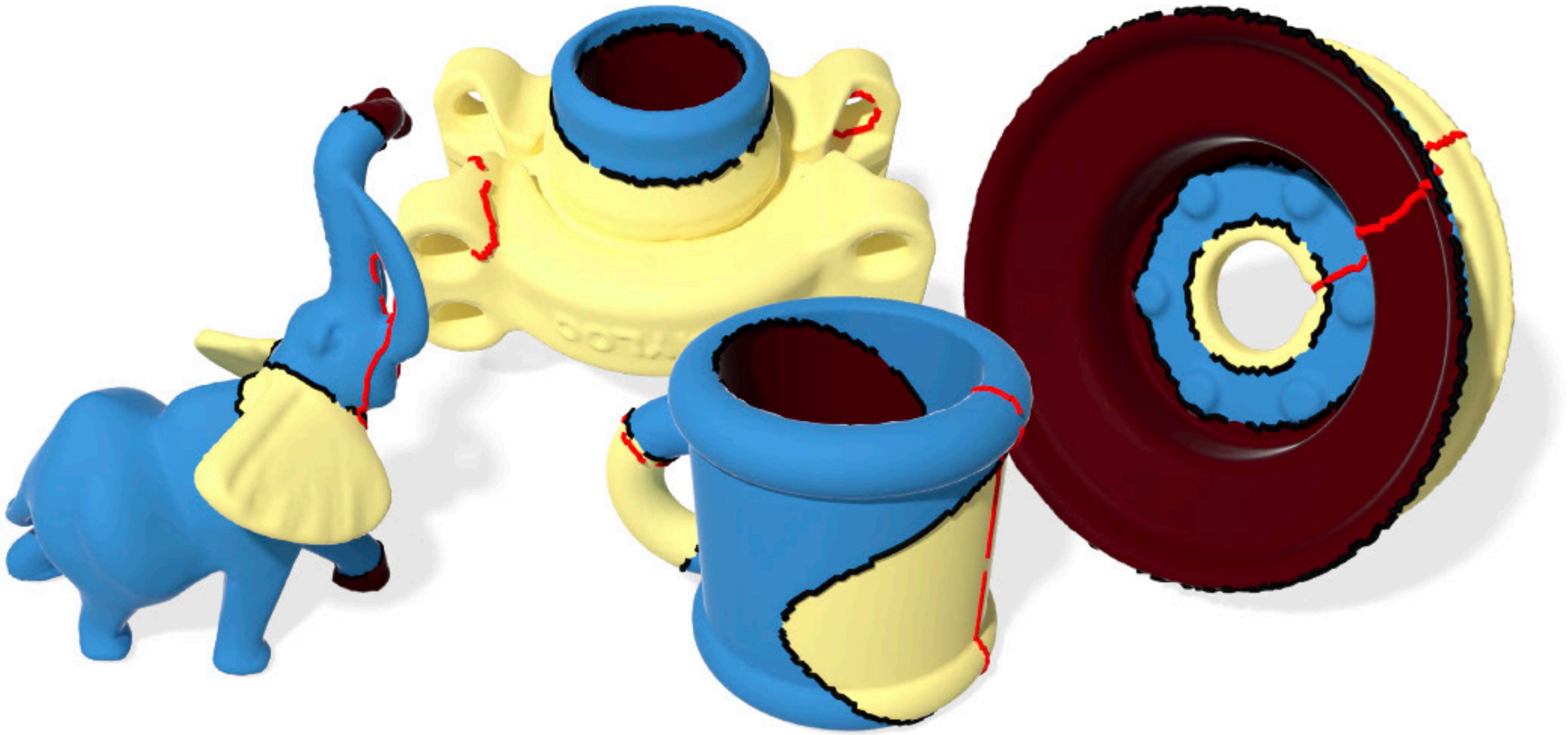
Evaluation



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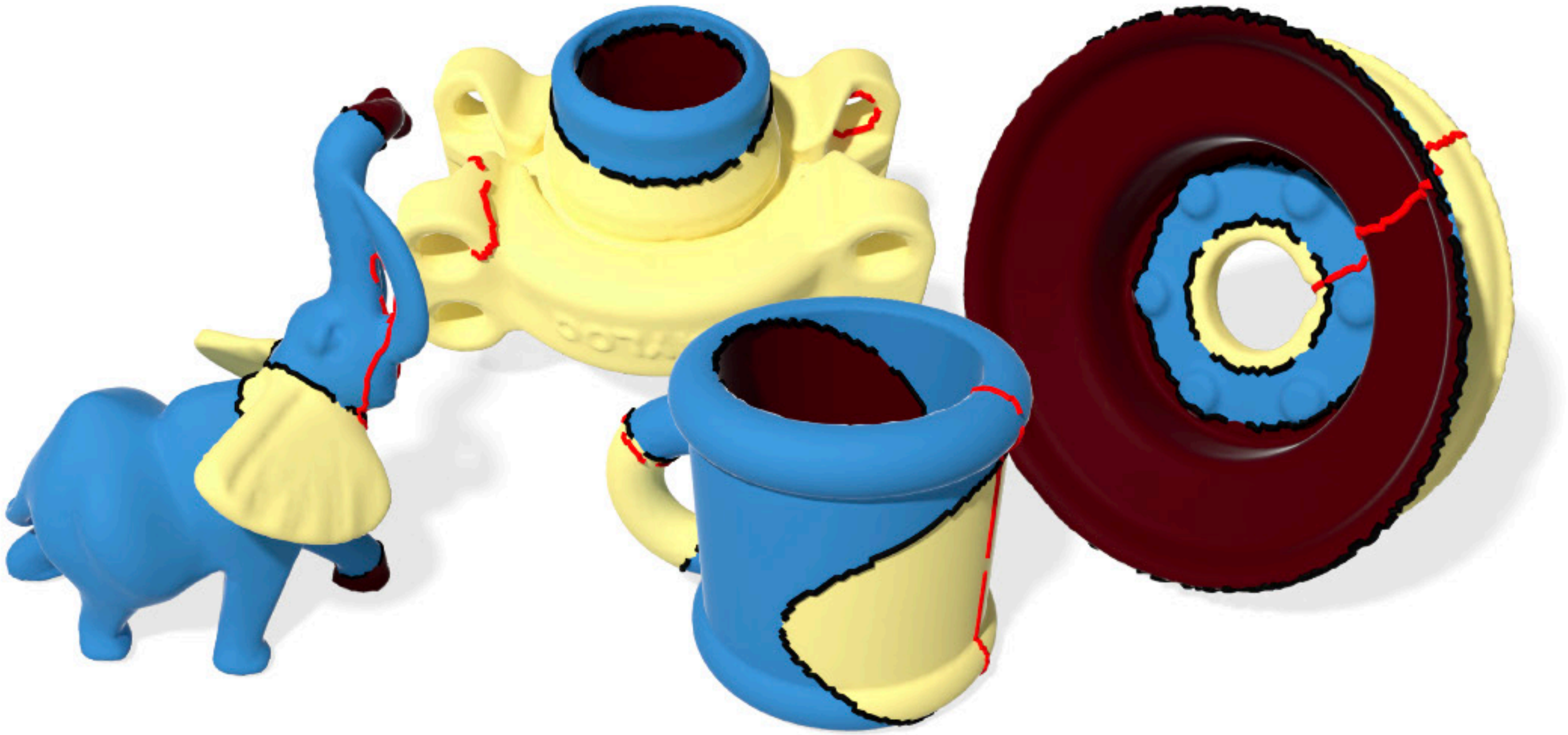

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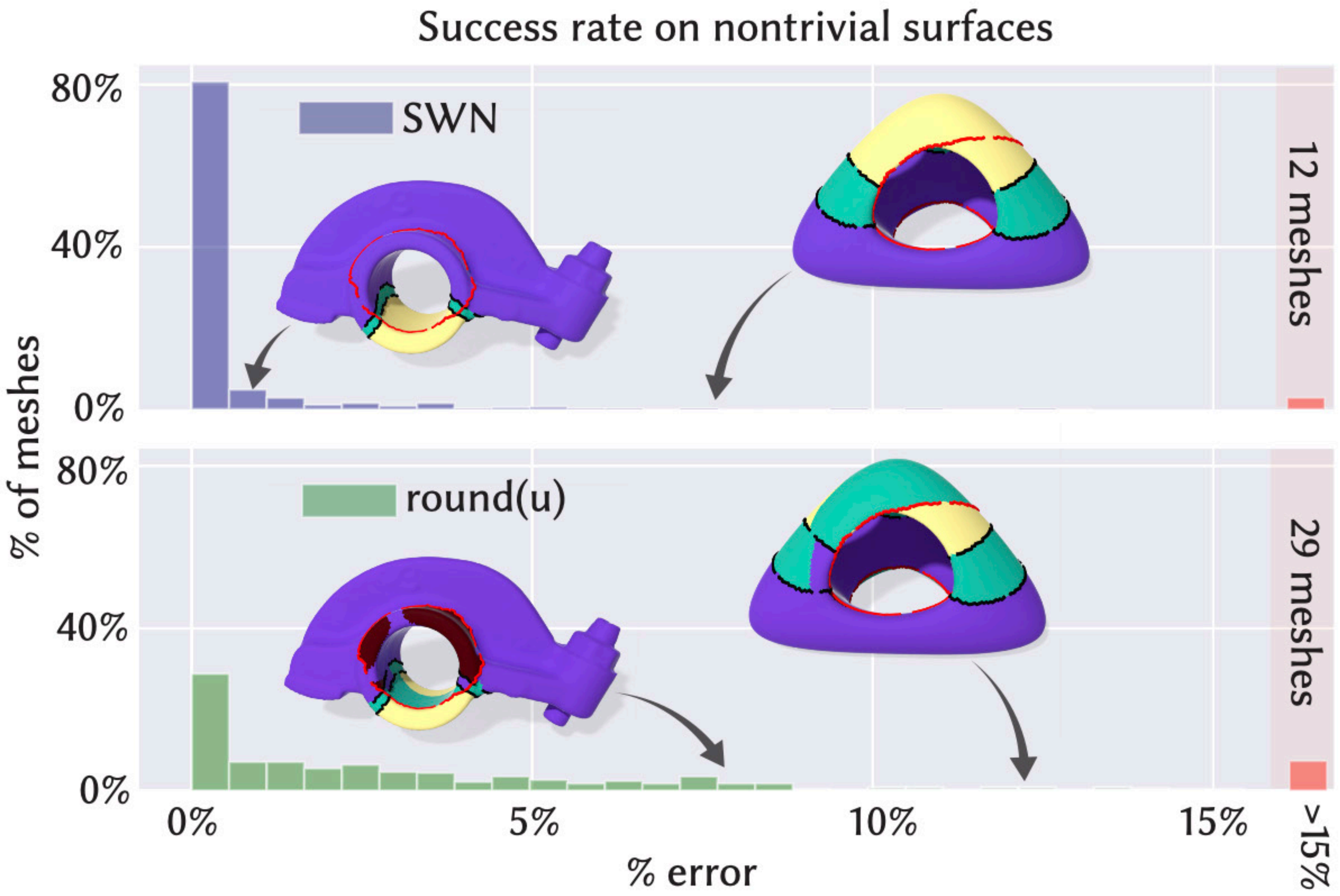
934 total test cases, 451 multiply-connected

Evaluation



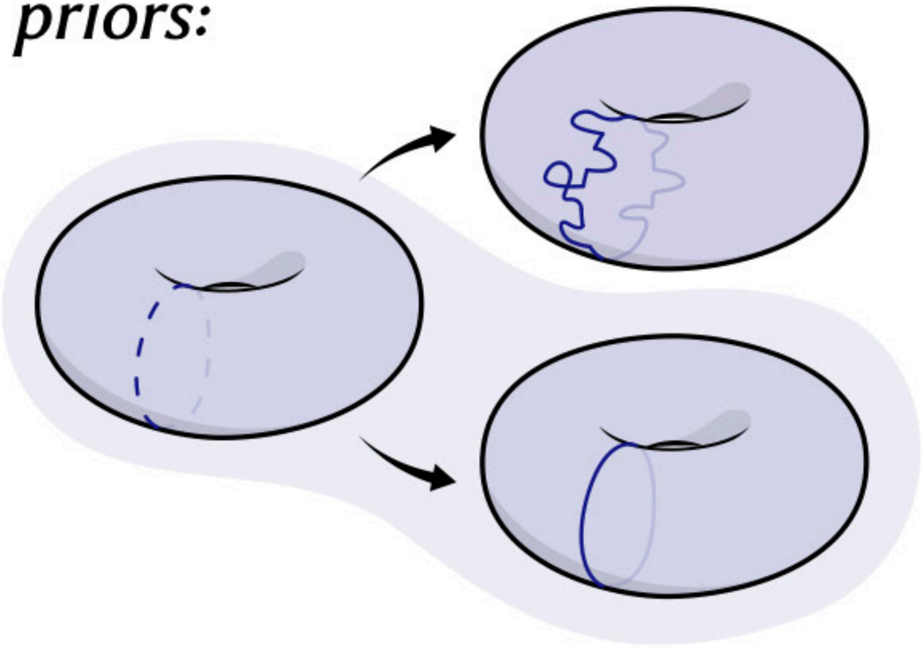
Ours

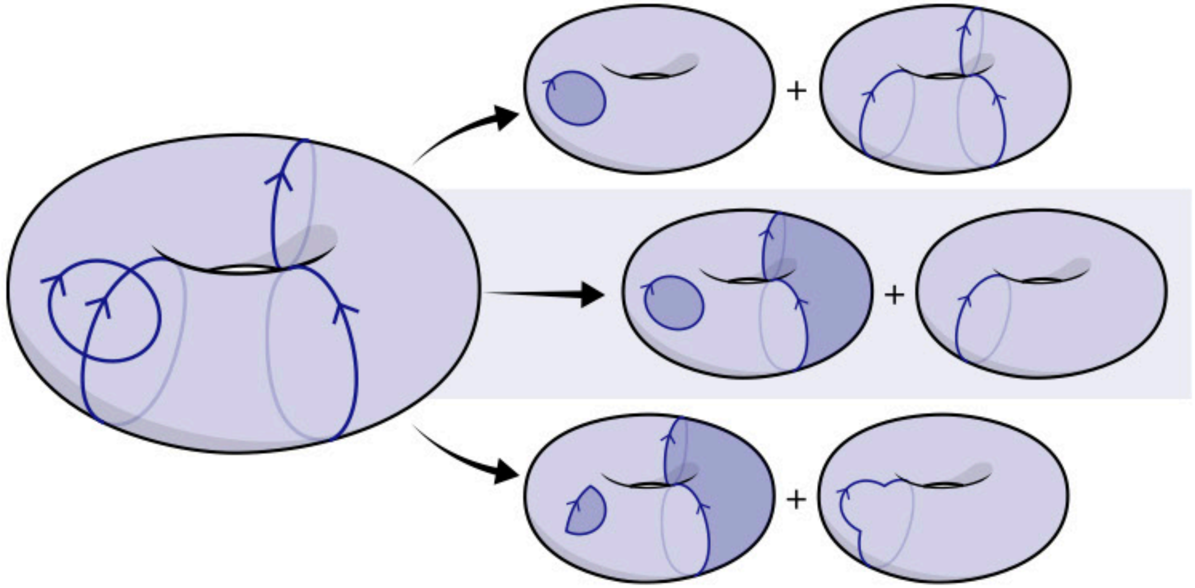
PSR/GWN

Much less accurate without homology processing

“Failure” cases

Recall shortest-length priors:





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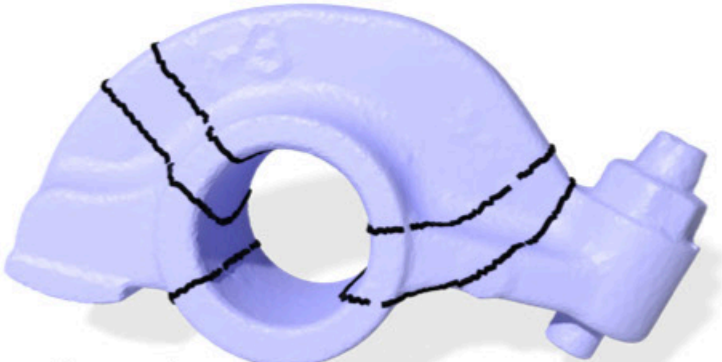
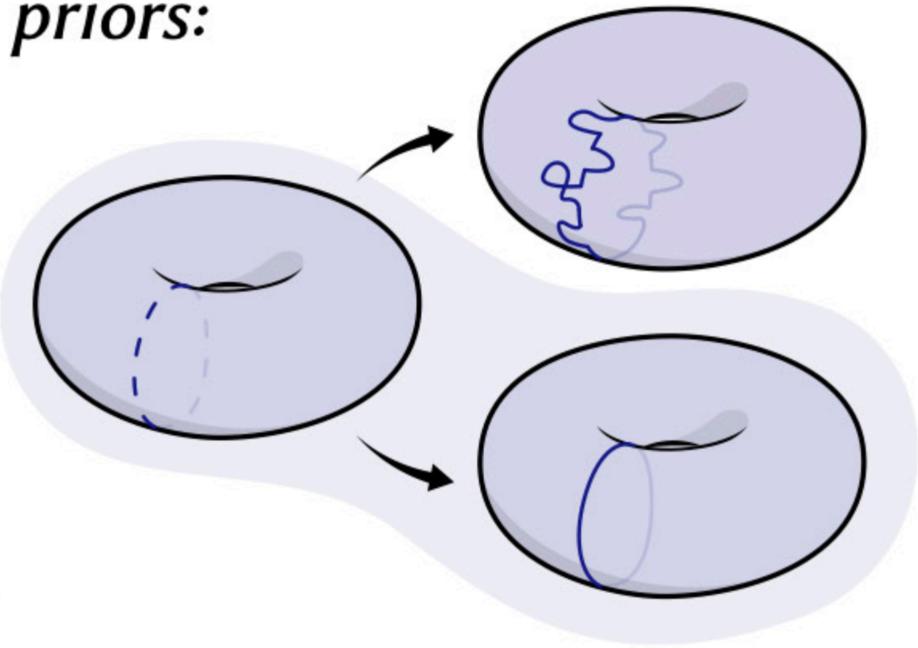
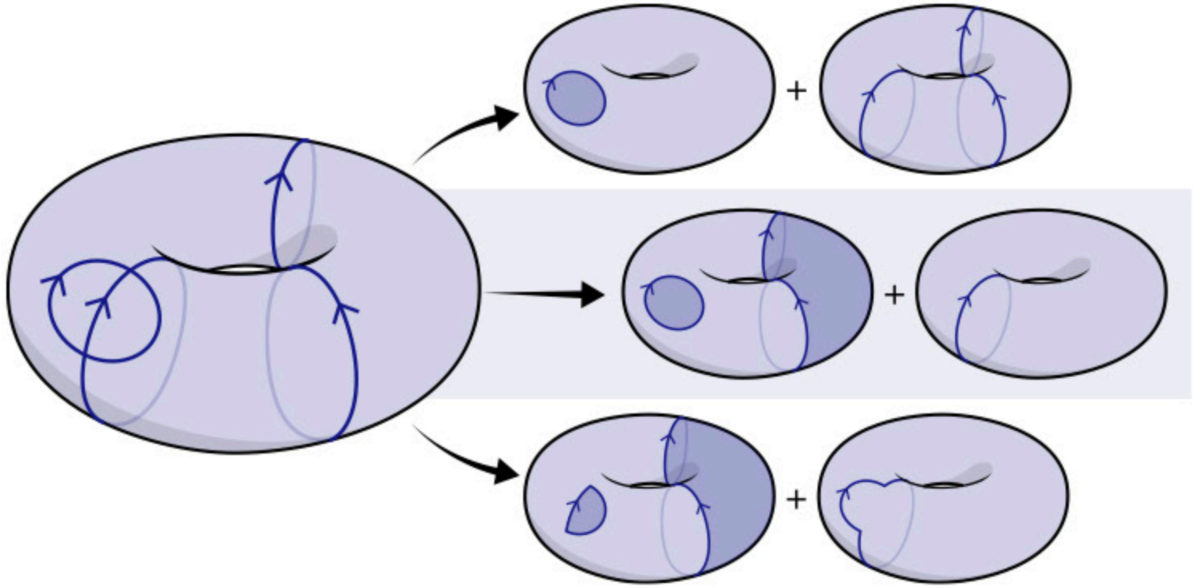


input





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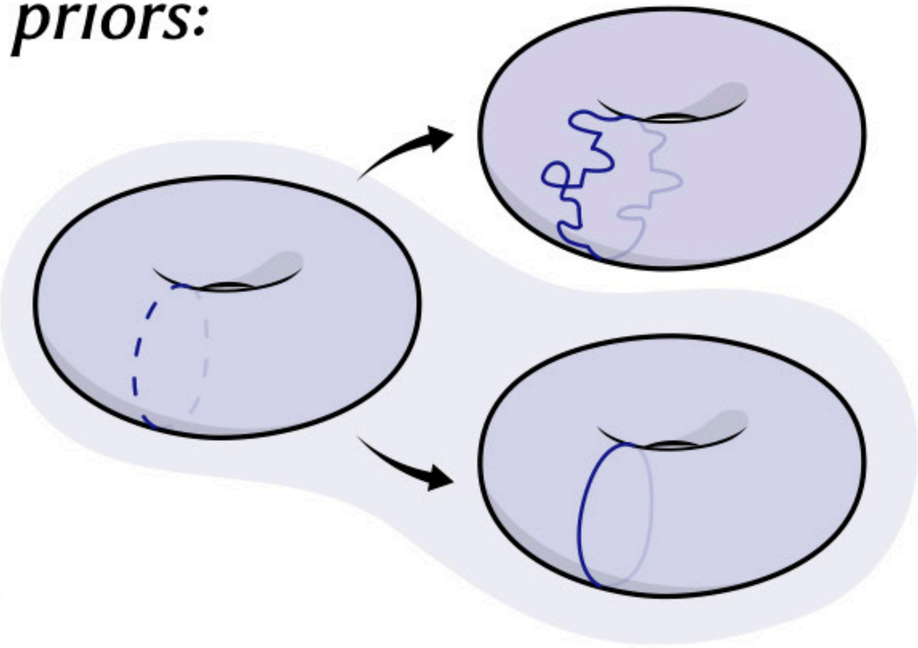

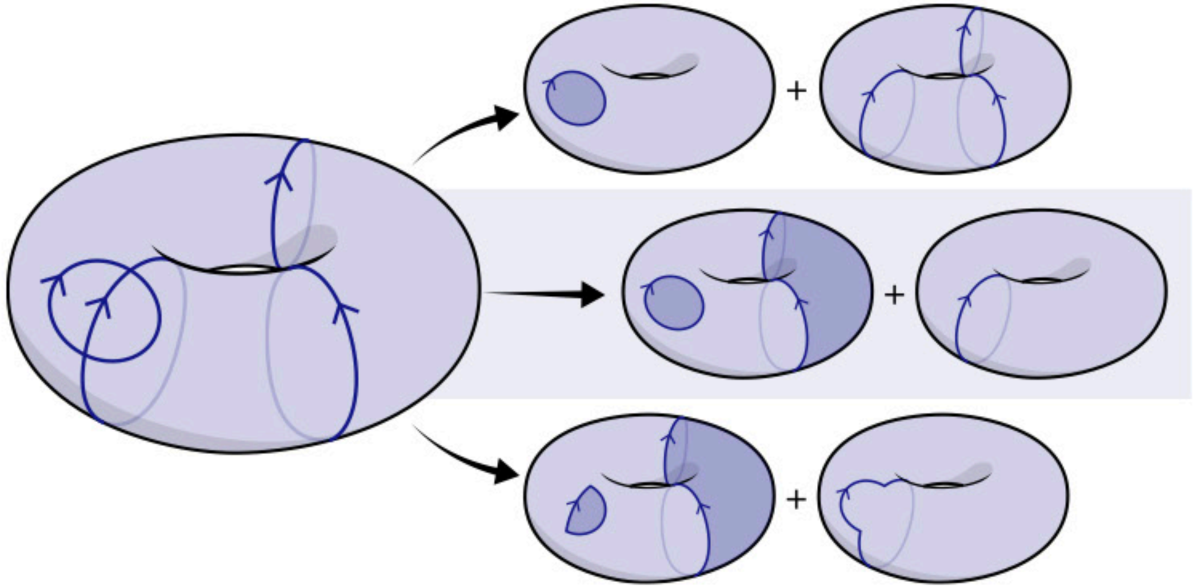

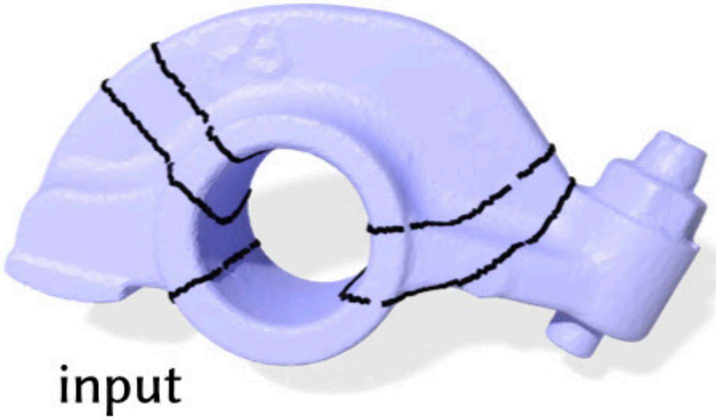

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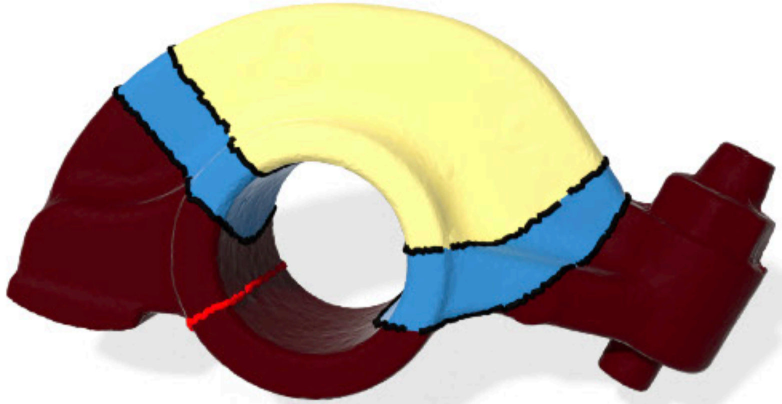
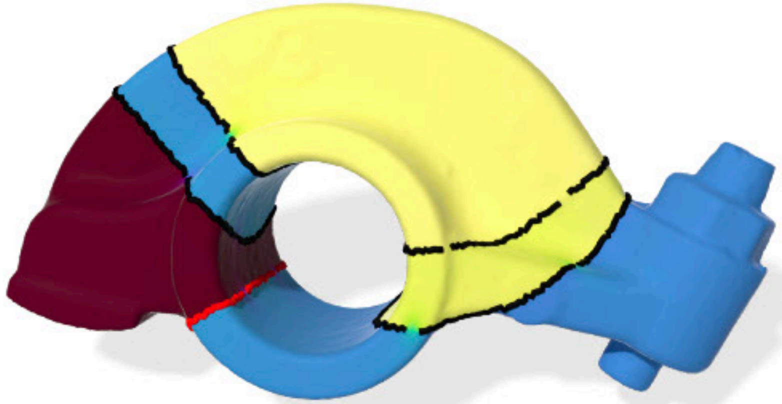


ground truth regions



SWN

“Failure” cases

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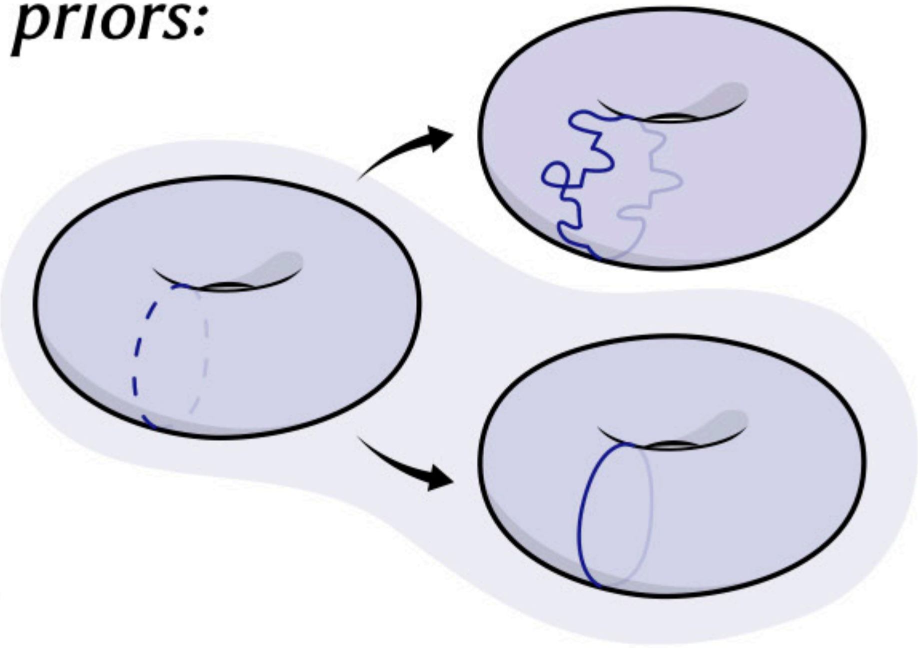
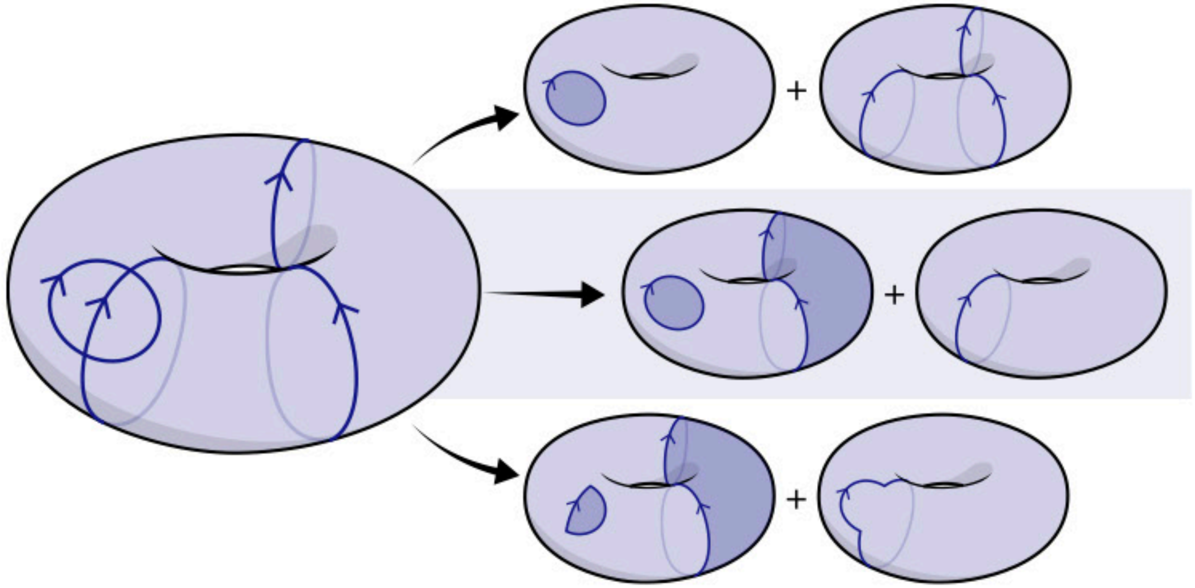
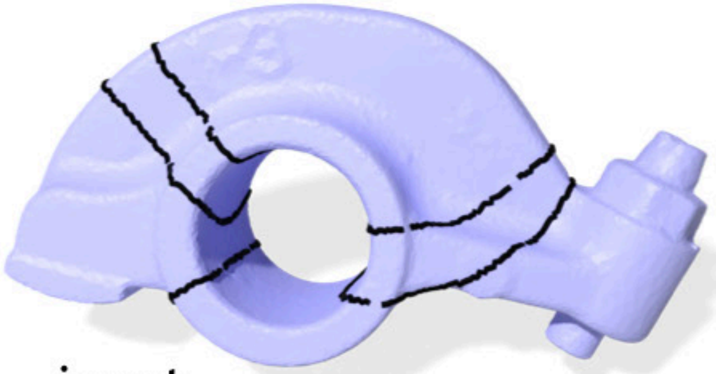
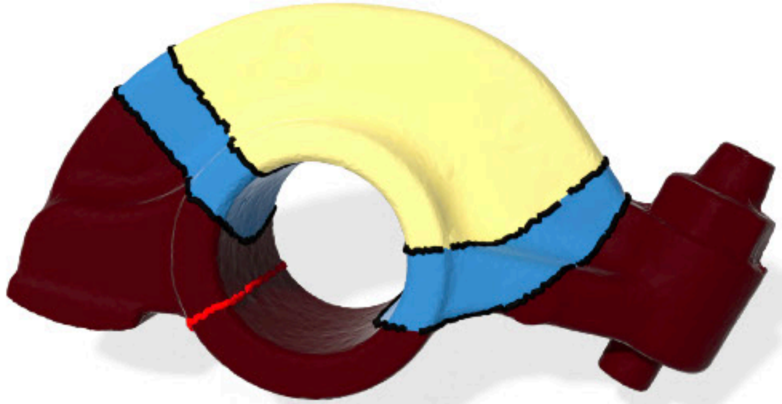






input



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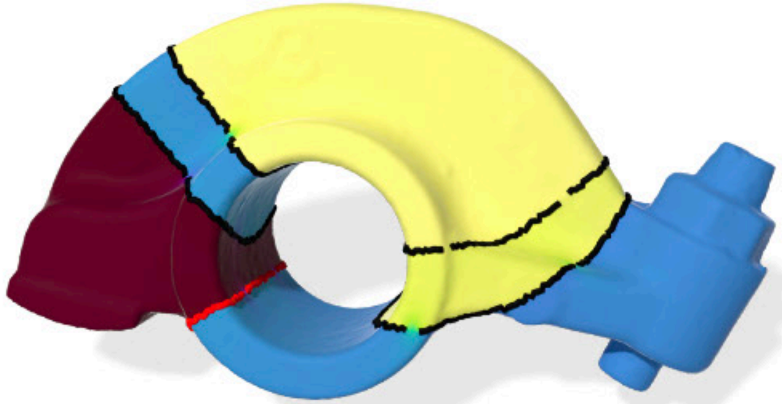
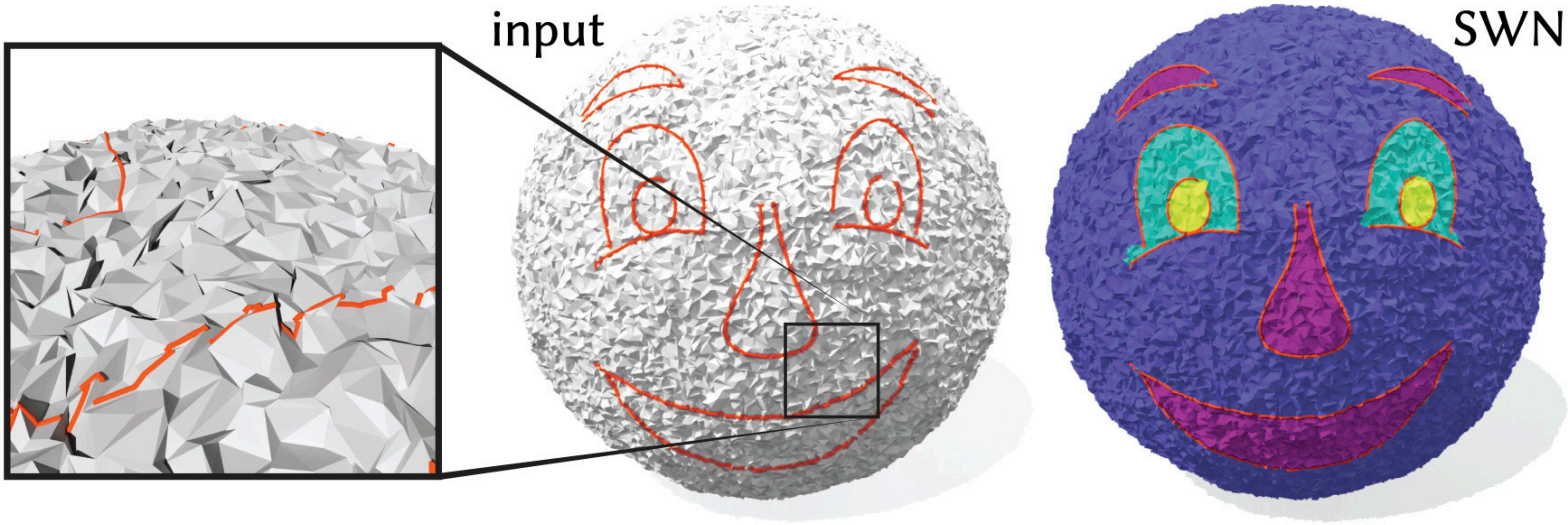


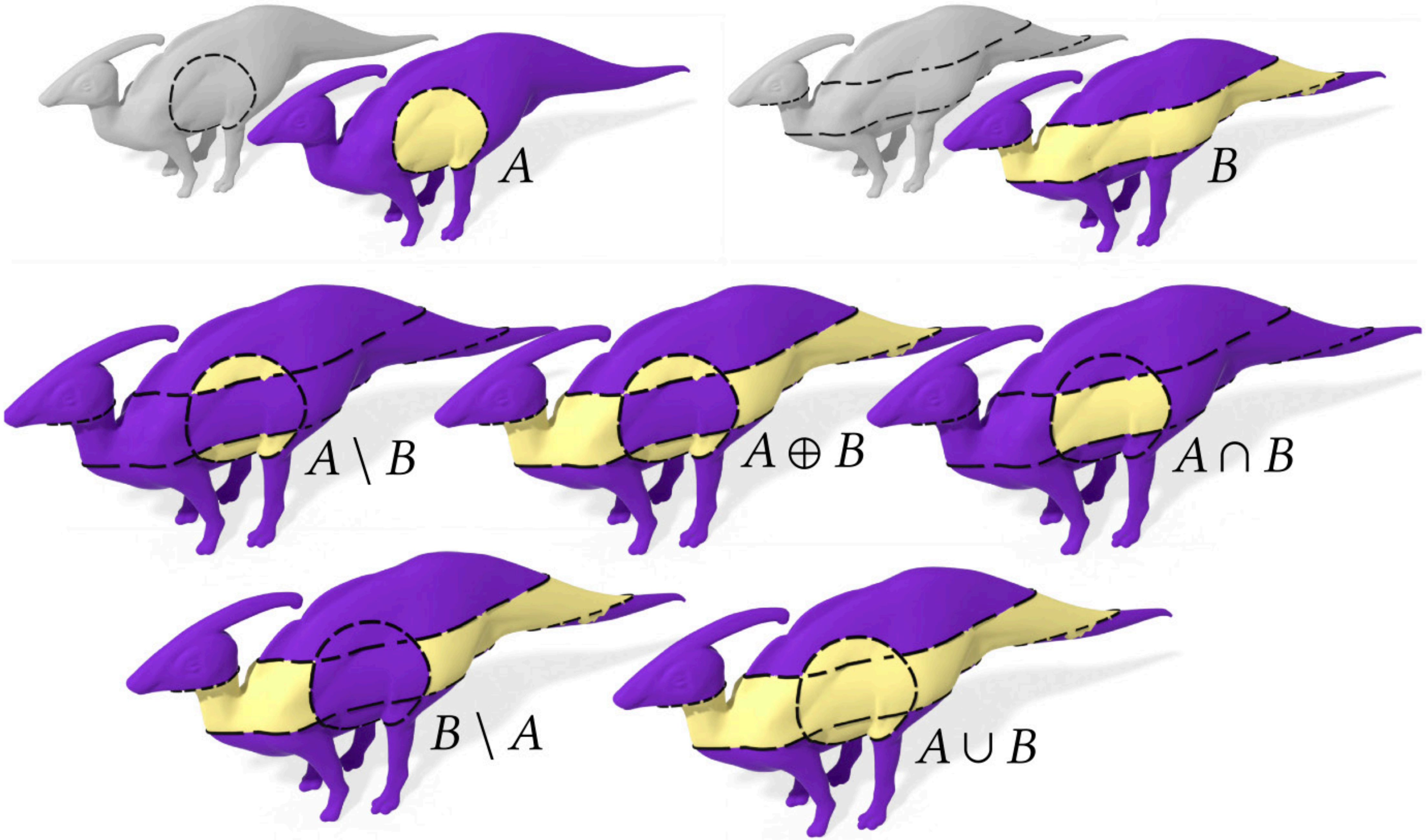
SWN

SWN still gives a valid curve decomposition!
(no single “right” answer.)

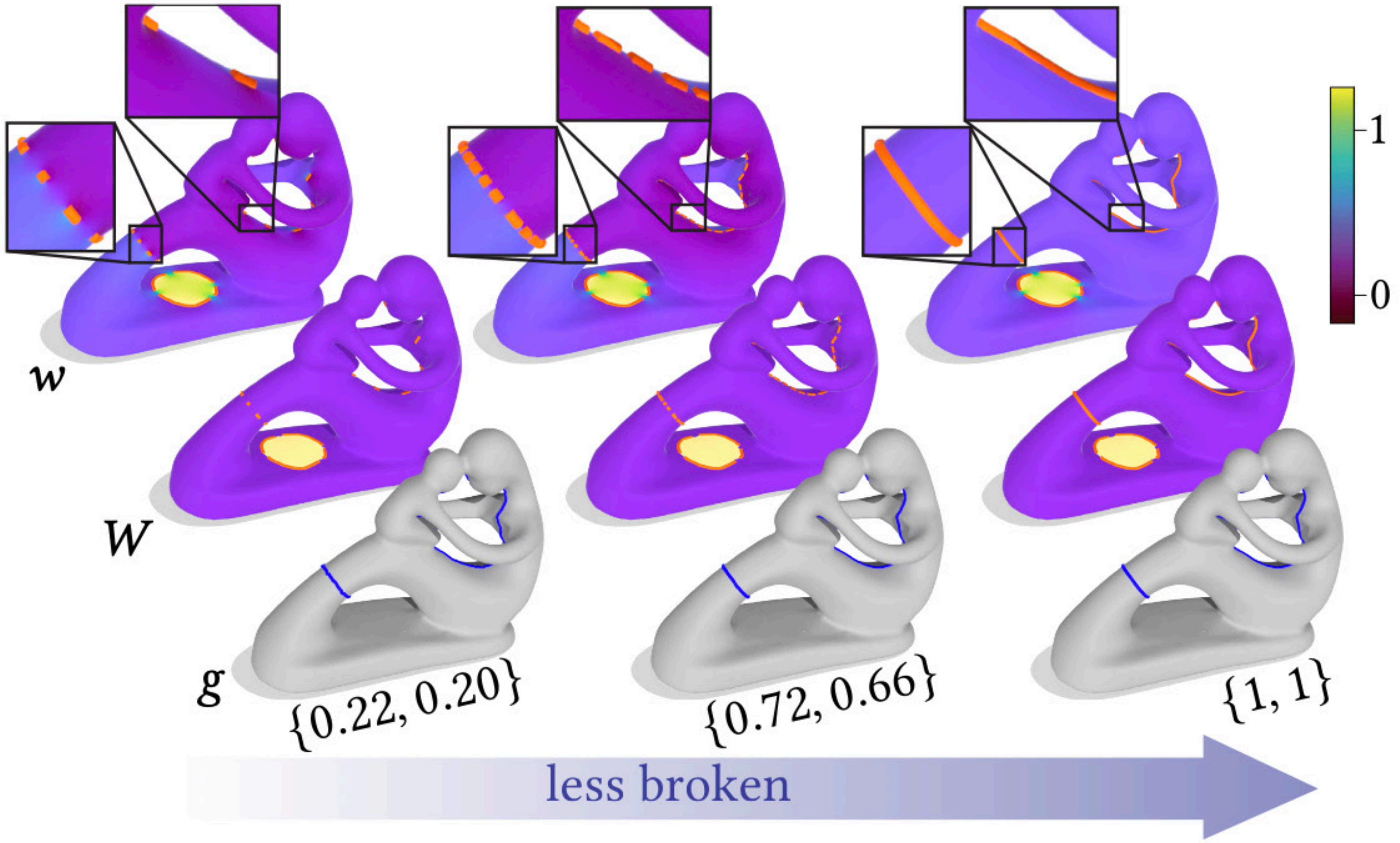
Surface sketching



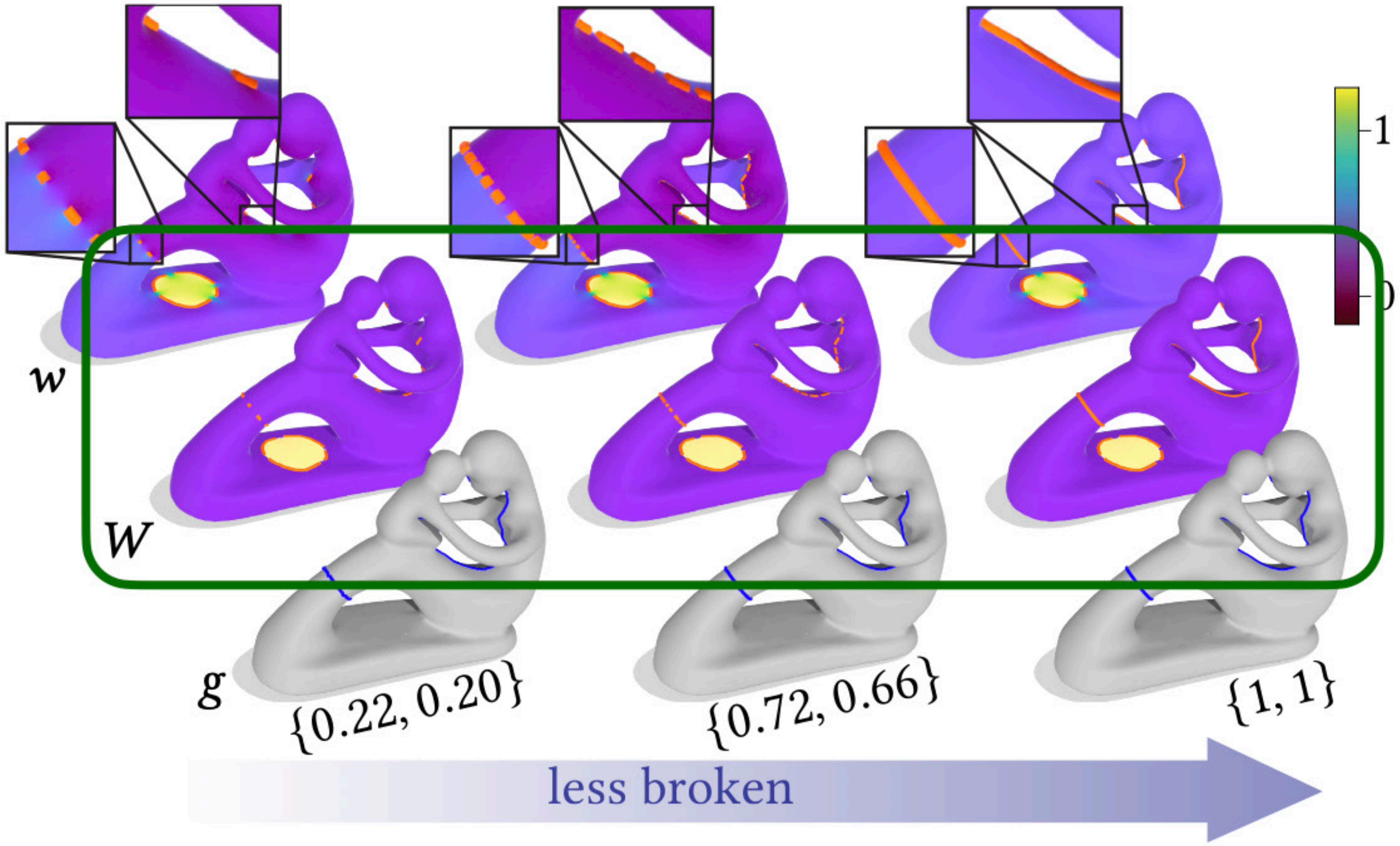
Booleans



Curve decomposition



Curve decomposition



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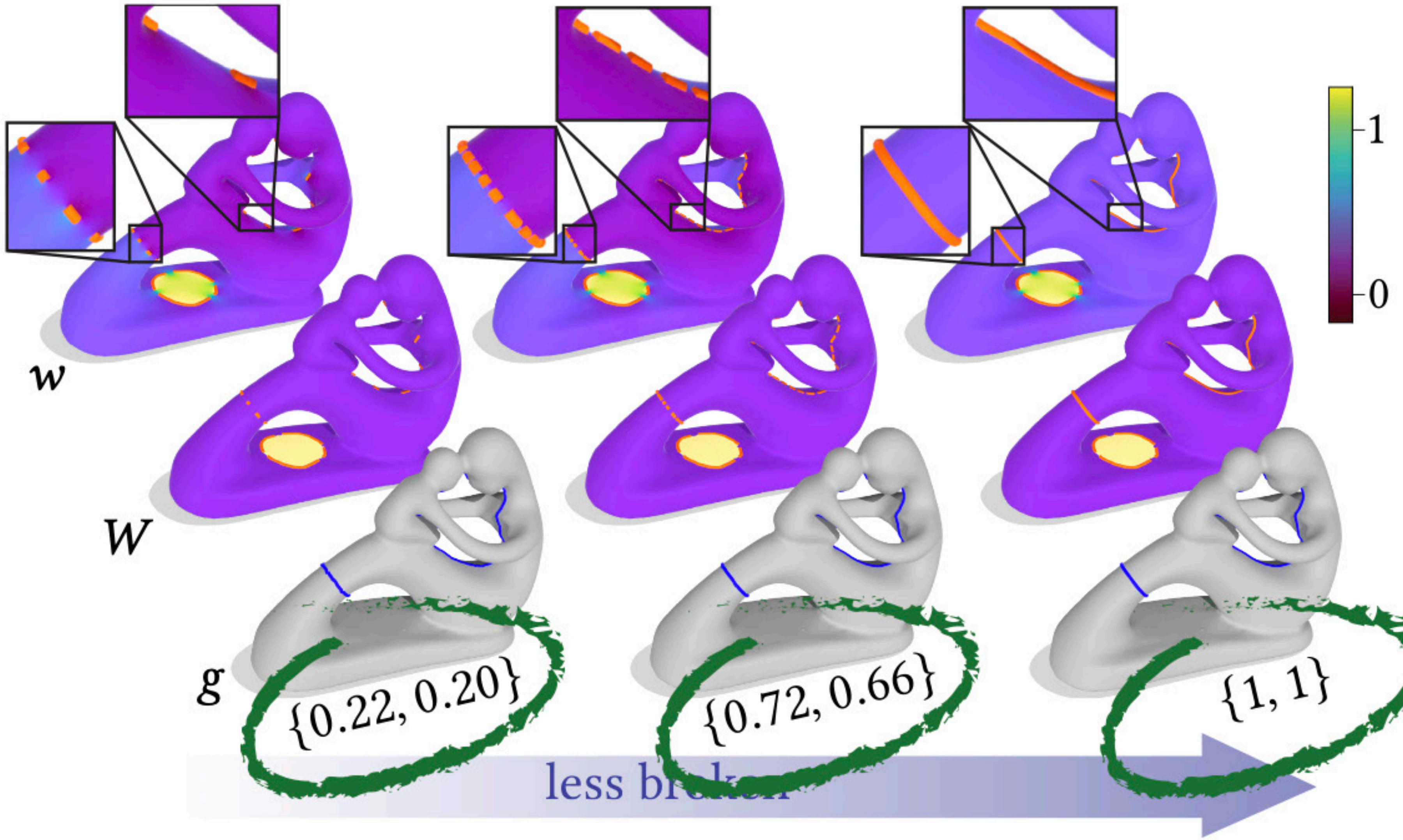
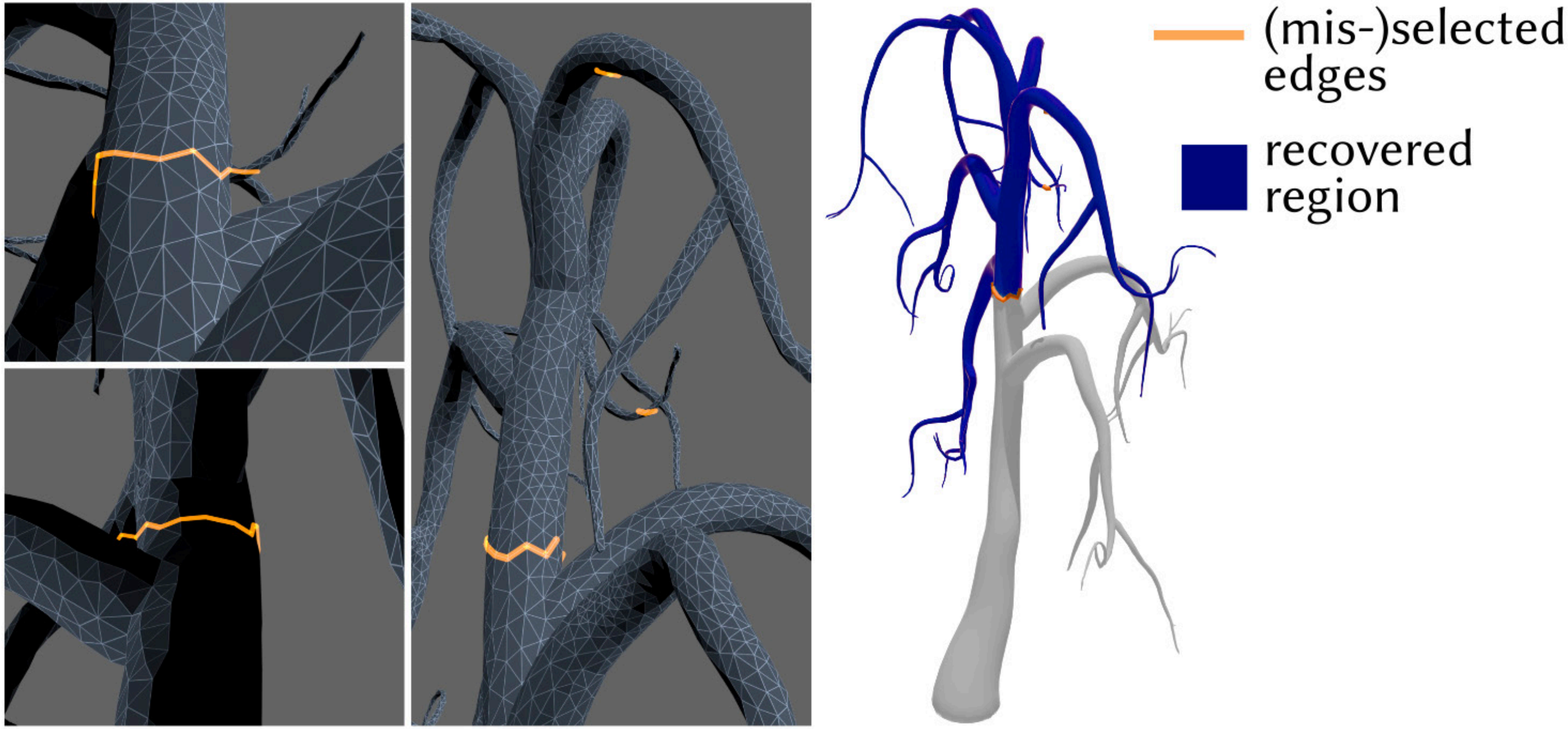


g values give SWN's confidence in nonbounding loops

Region selection — robustness

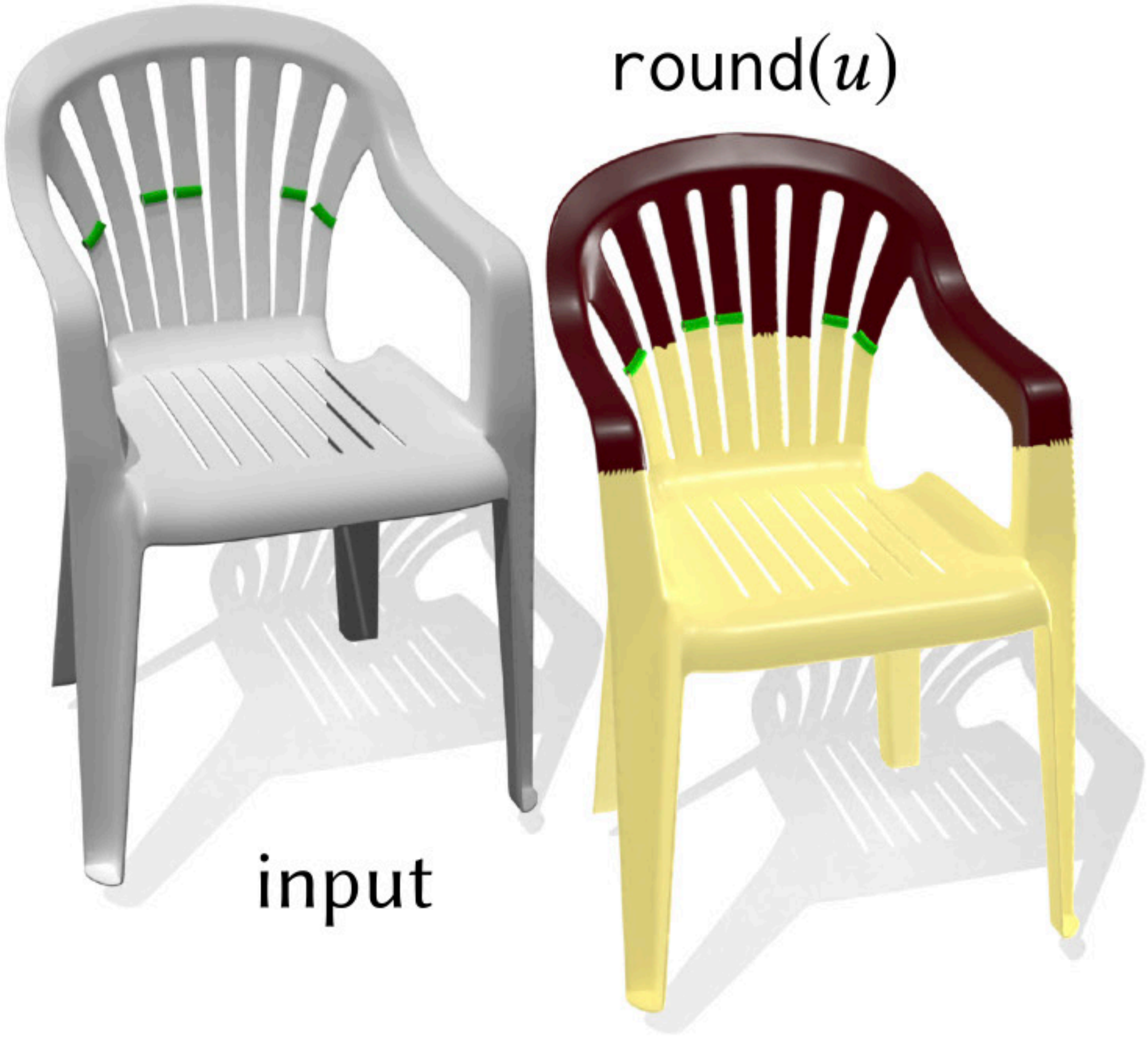


Region selection — automatic completion

input

Region selection — automatic completion

using "ghost loops" to *aid* segmentation



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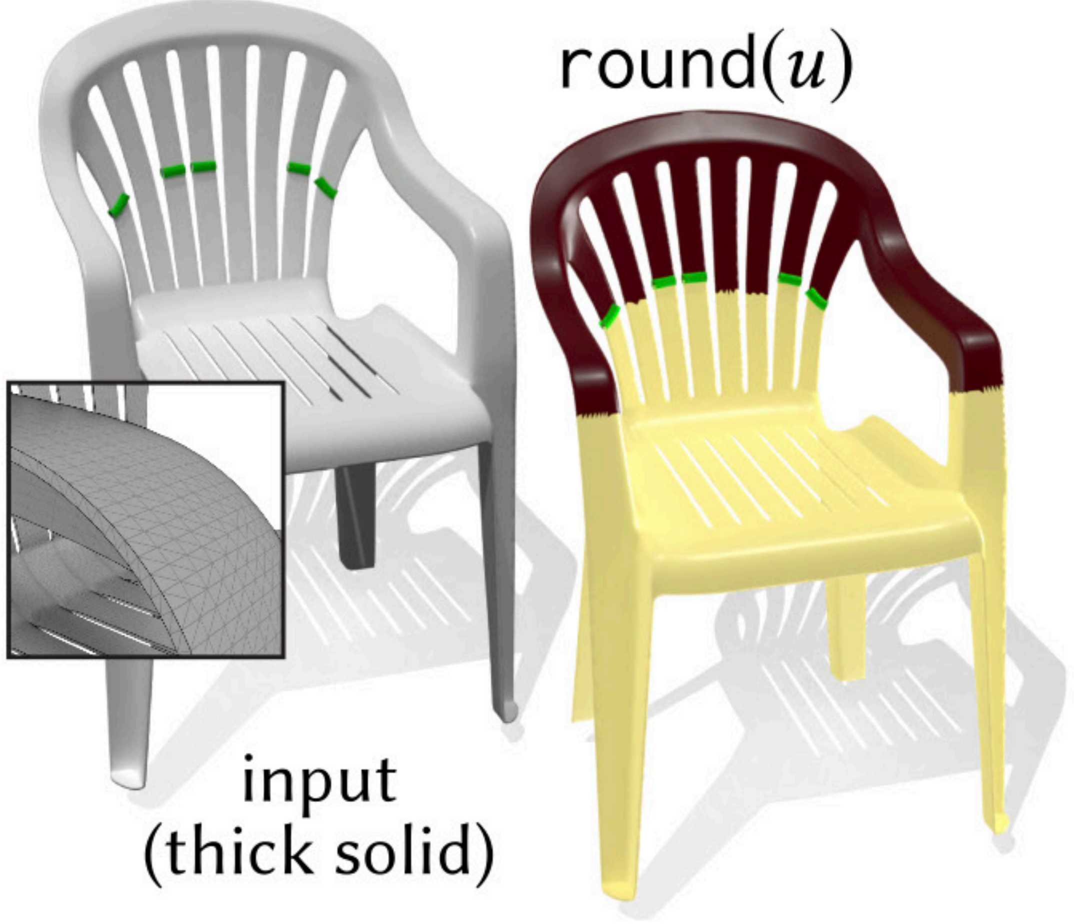

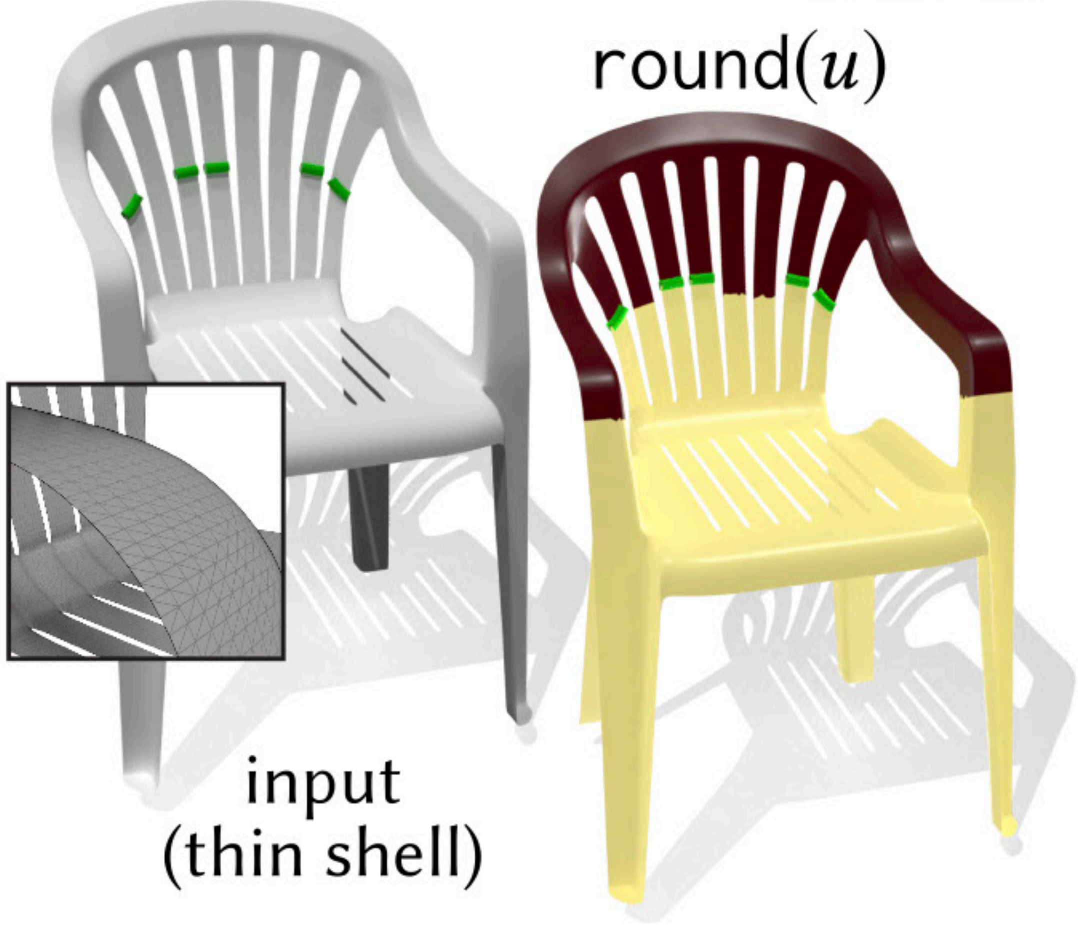

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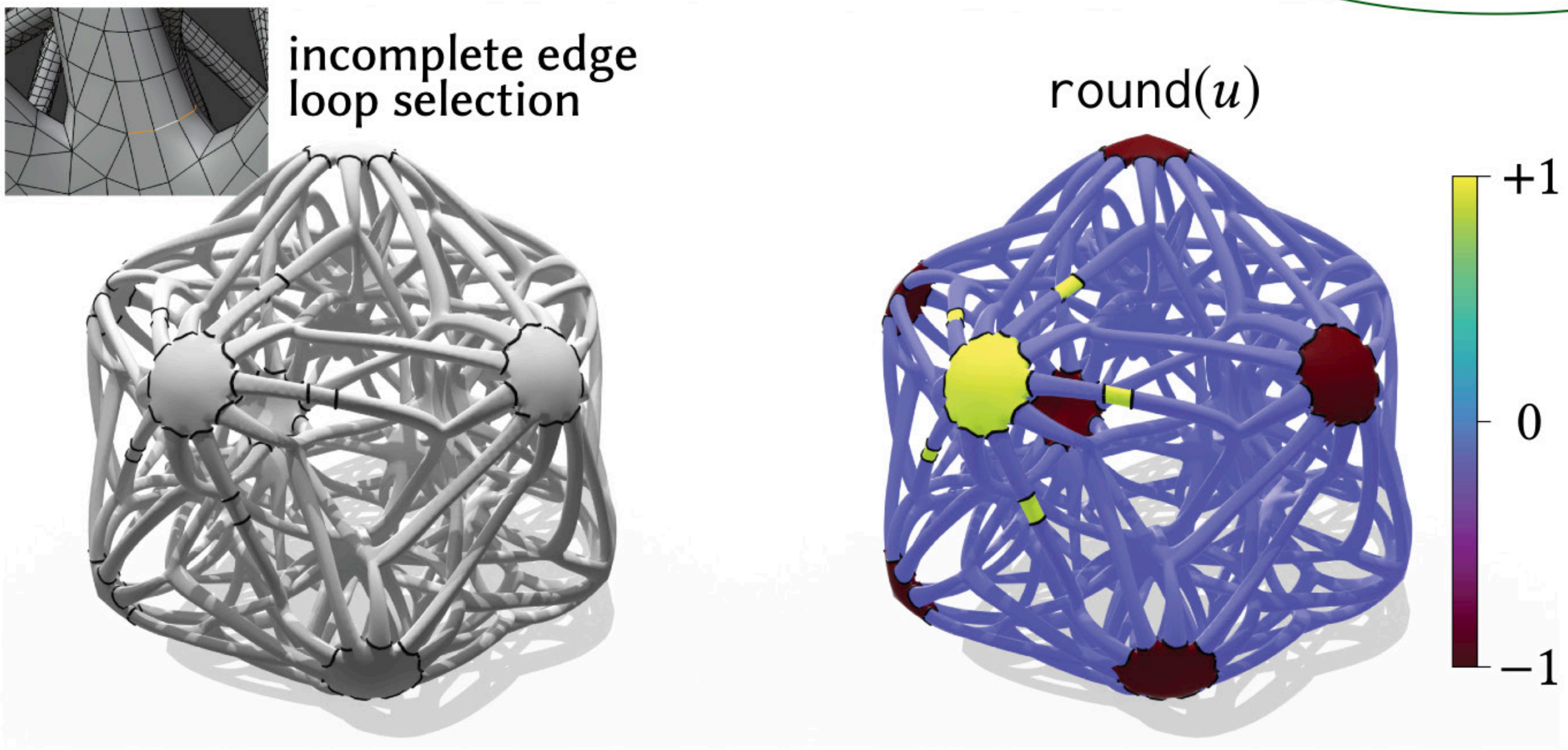
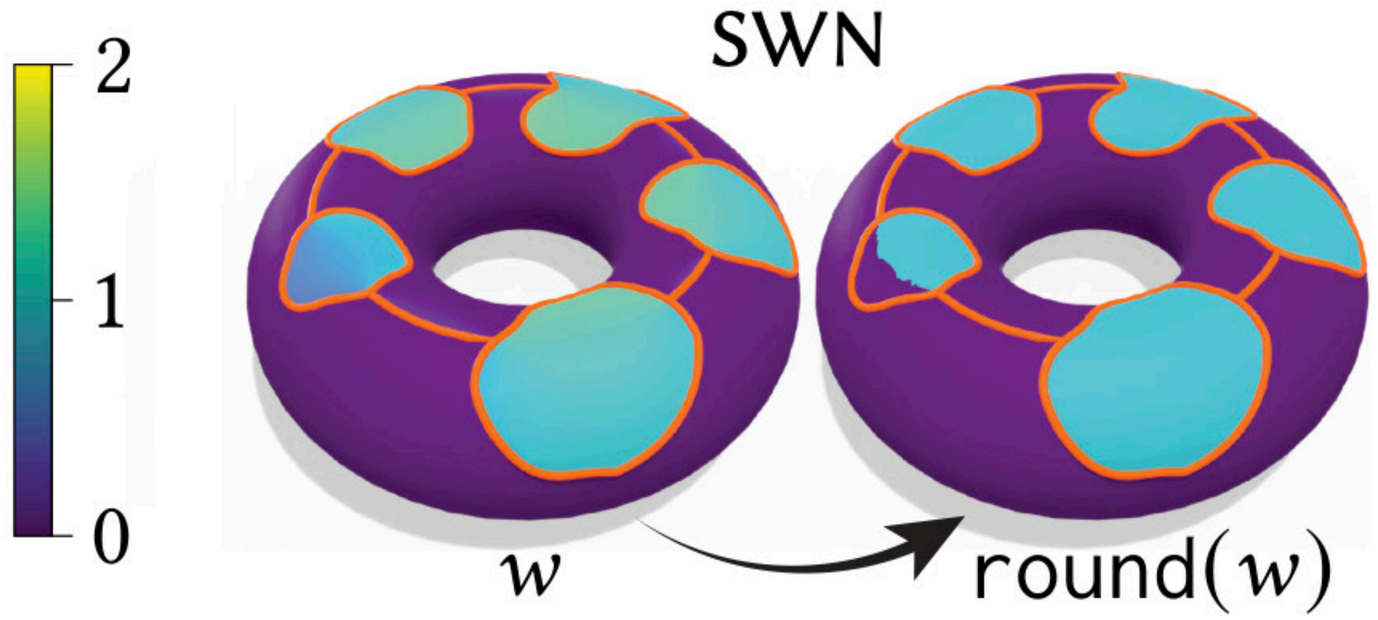


LIMITATIONS & FUTURE WORK

Contouring is sometimes counterintuitive



Contouring is sometimes counterintuitive

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Euclidean winding number also struggles

Jacobson et al. suggest a graphcut algorithm for contouring:

Robust Inside-Outside Segmentation using Generalized Winding Numbers
Jacobson, Kavan, Sorkine-Hornung (2013)

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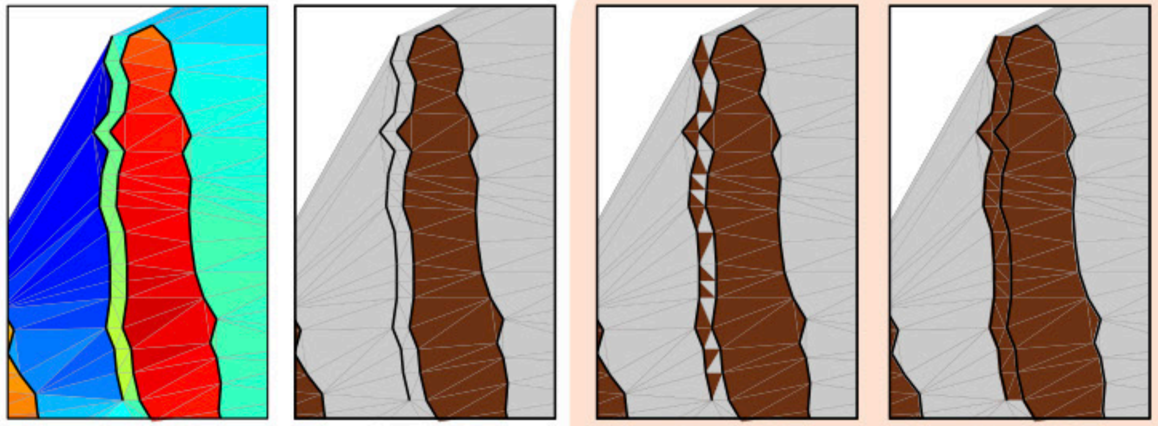

GWN

naive contouring

graphcut

Robust Inside-Outside Segmentation using Generalized Winding Numbers
Jacobson, Kavan, Sorkine-Hornung (2013)

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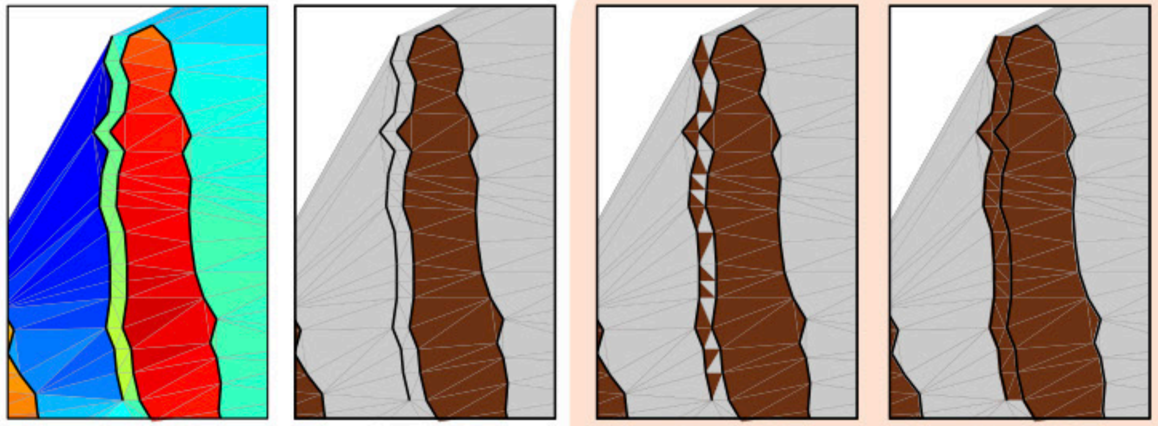
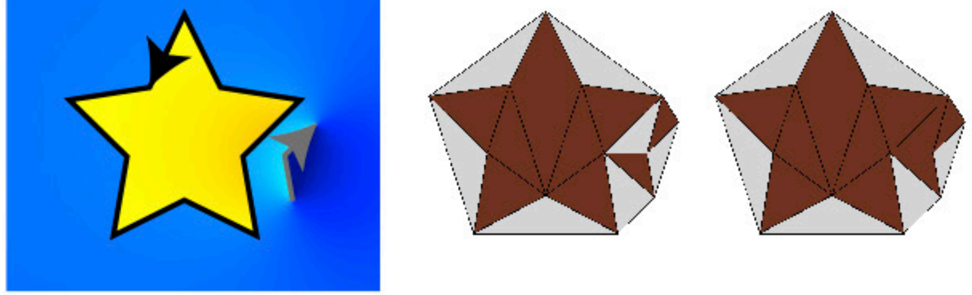


GWN naive contouring graphcut



GWN contoured

Still not perfect!

Robust Inside-Outside Segmentation using Generalized Winding Numbers
Jacobson, Kavan, Sorkine-Hornung (2013)

Unexpected nonbounding loops

Recall objective function: $\min_{v:\ M \to \mathbb{R}} \int |\text{the jumps not across } \Gamma| + \varepsilon \int |\text{the jumps across } \Gamma|$

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encourage jumps across Γ

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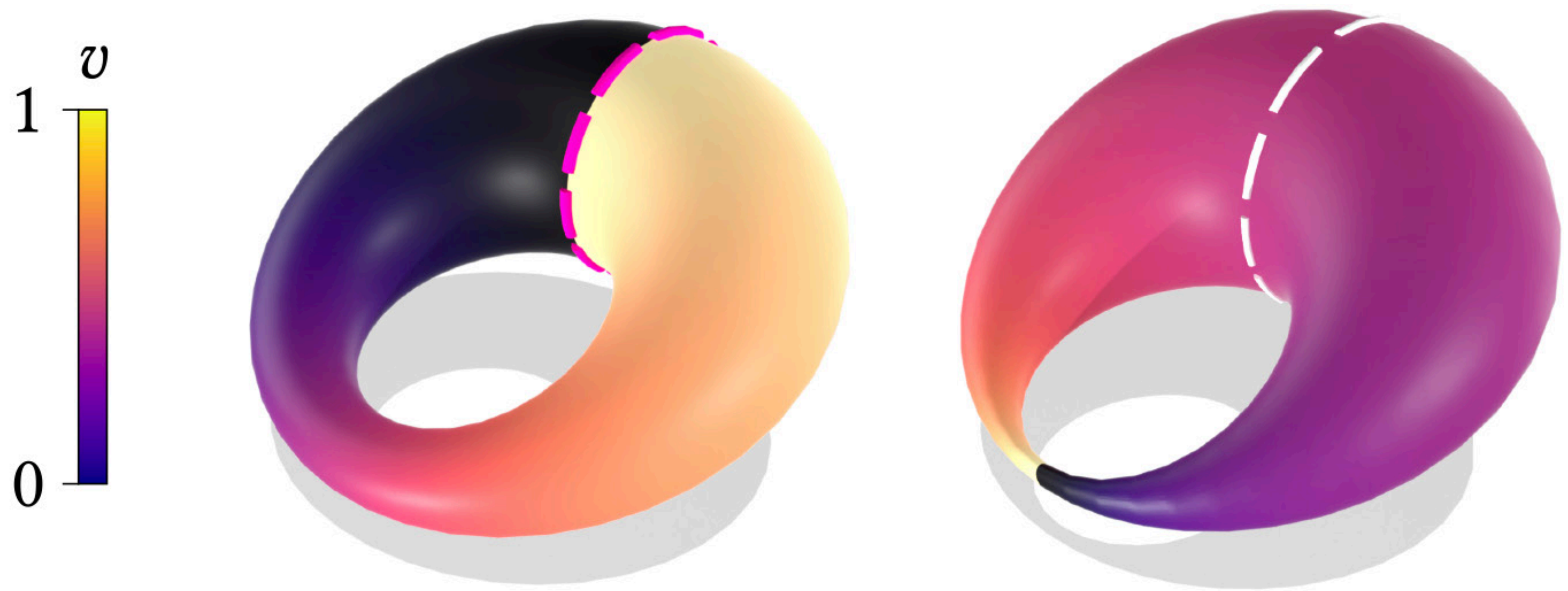

Unexpected nonbounding loops

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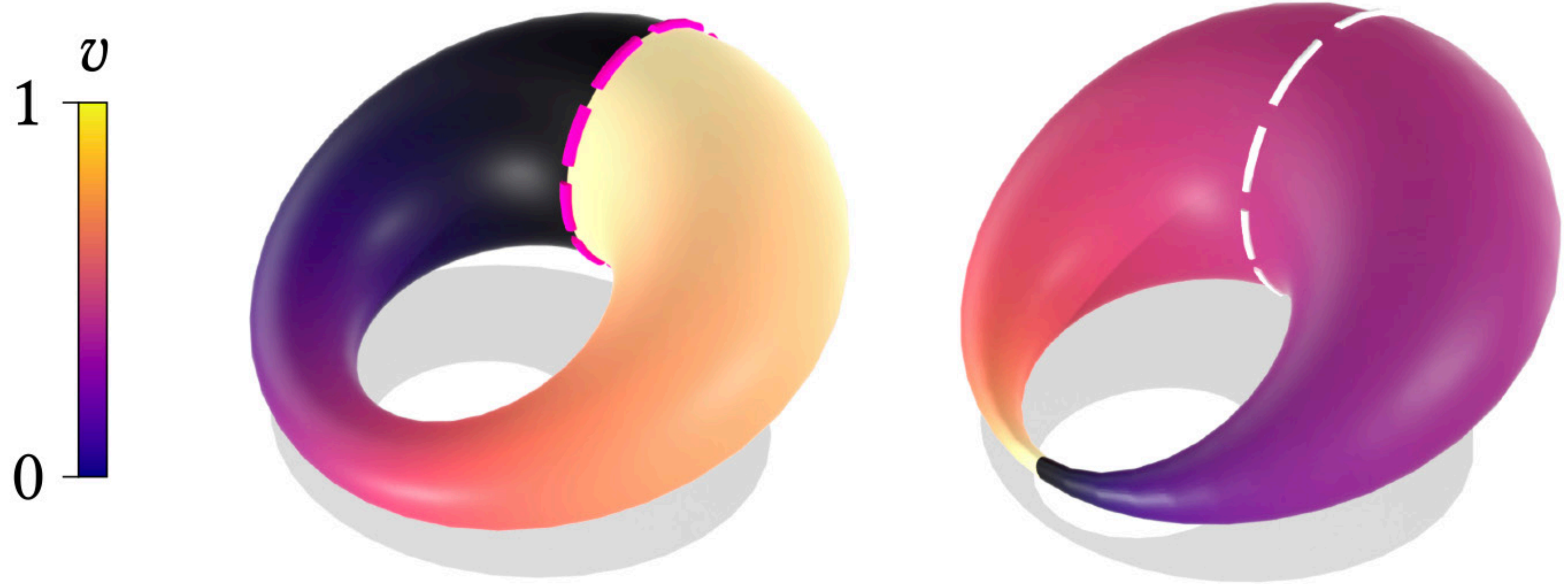

Can always adversarially increase a handle's taper.

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But given a *fixed* mesh, we will recover the correct solution as gaps → 0 (and appropriate choice of ε).

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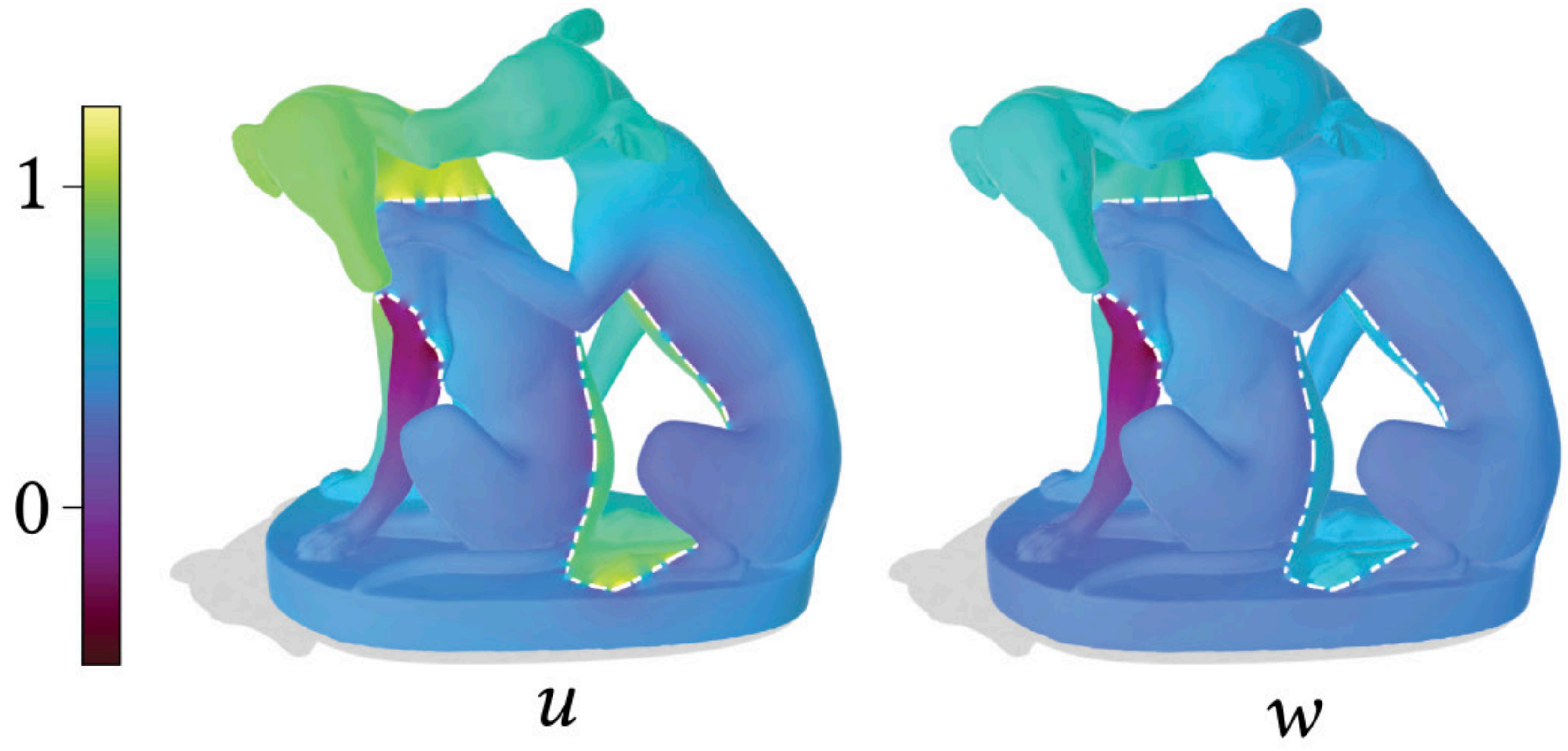
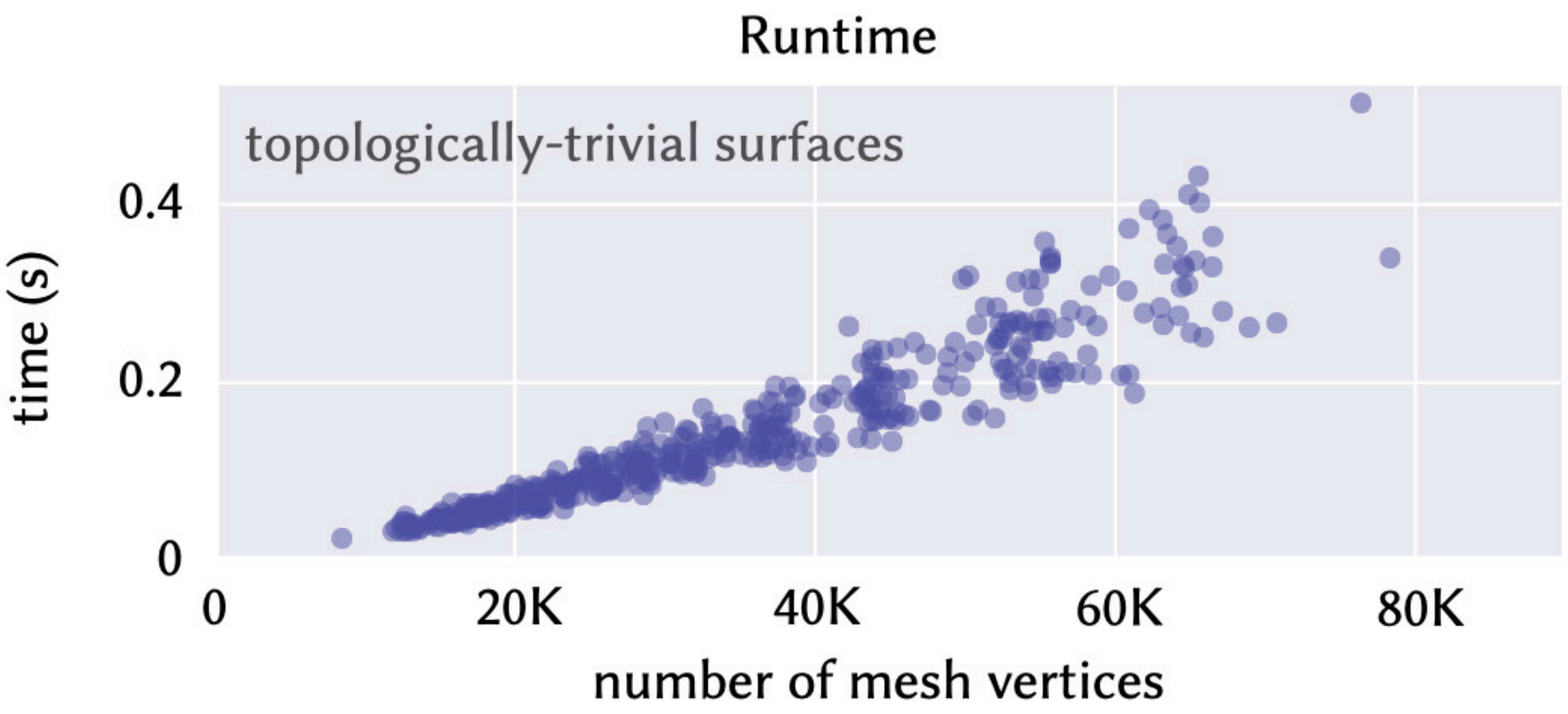


Performance



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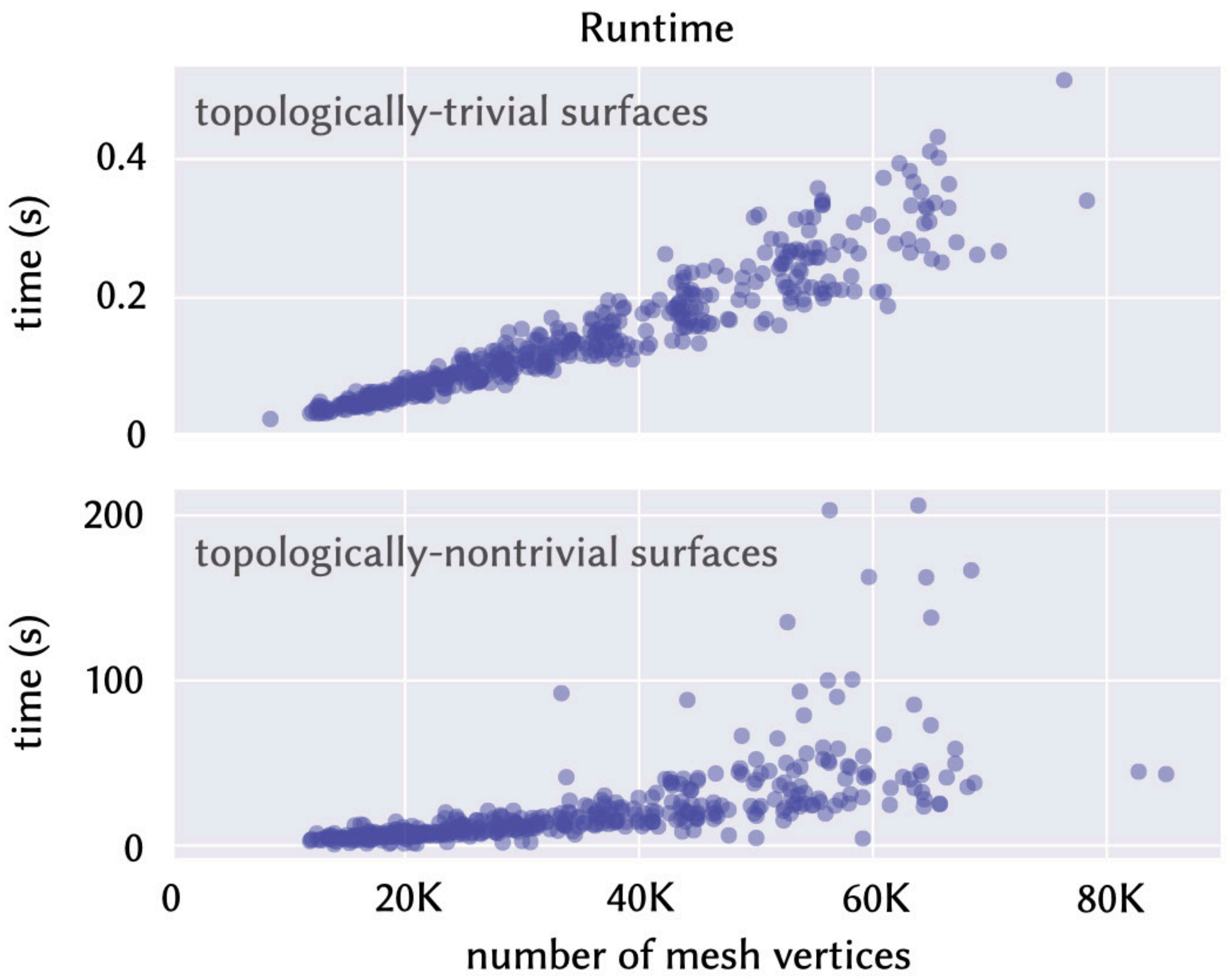

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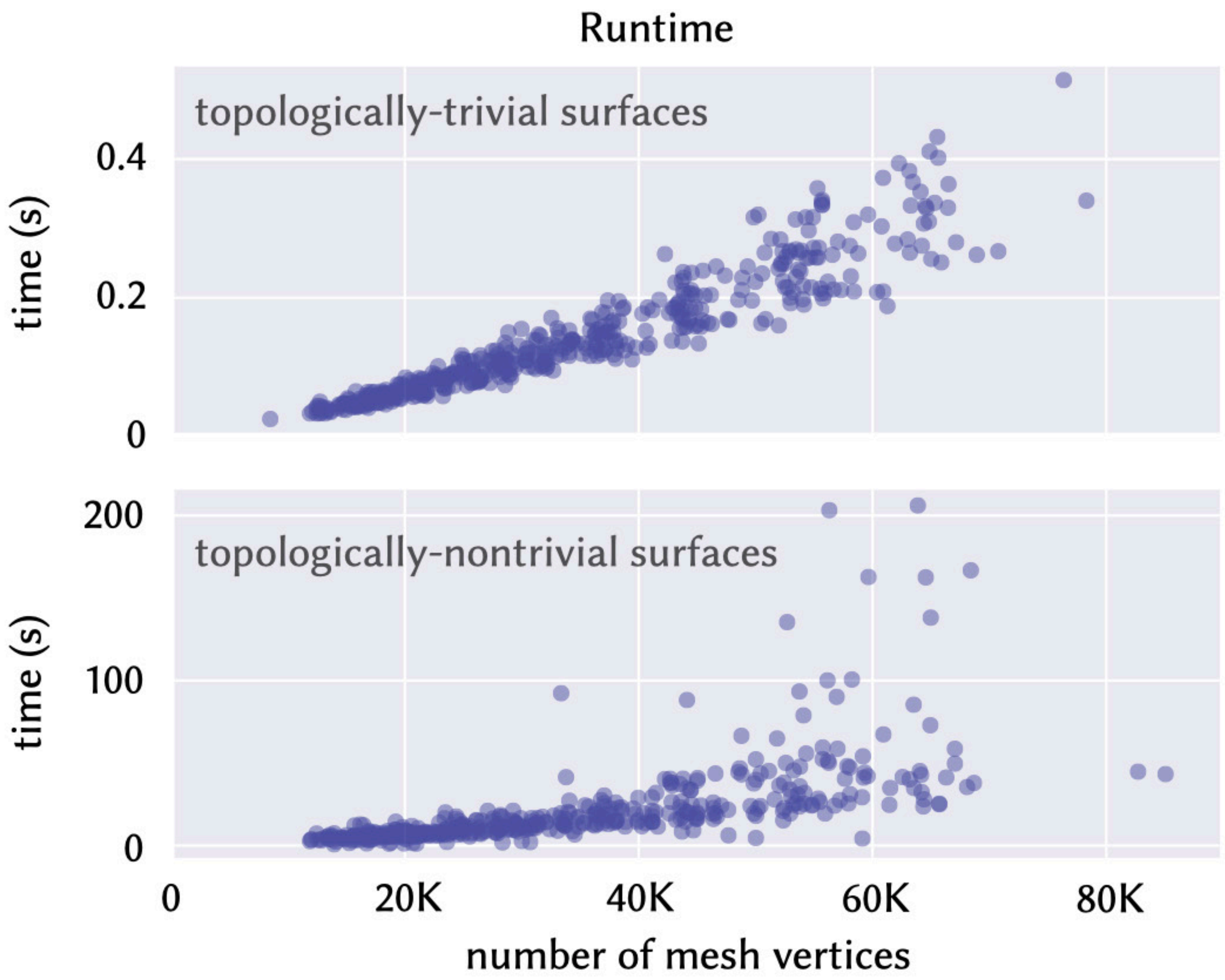

Computation dominated by linear program.

Reduced-size linear program

Current implementation simultaneously optimizes both jump locus *and* jump magnitudes.

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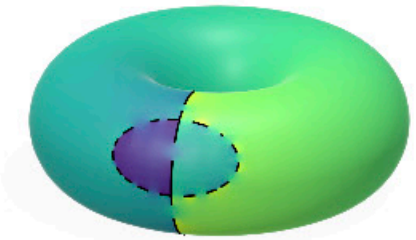

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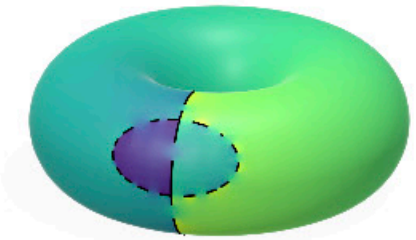

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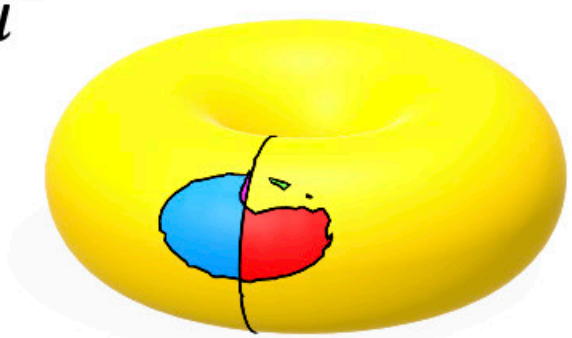

Dijkstra-completed curve

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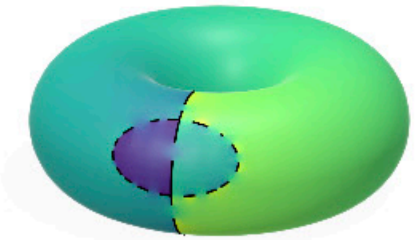

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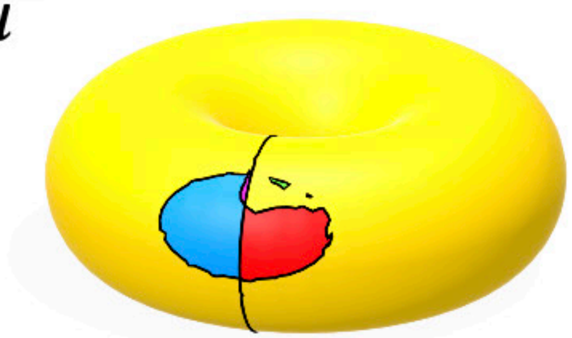
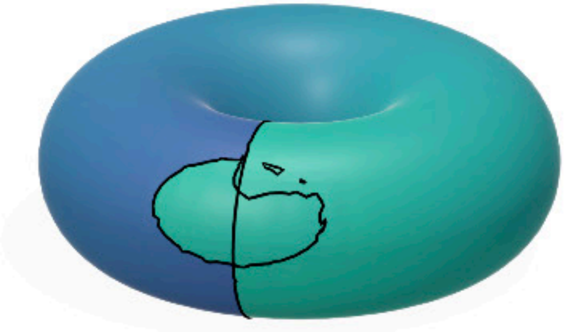


Dijkstra-completed curve

Integrate γ locally within each connected component



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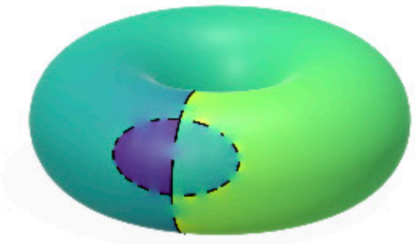

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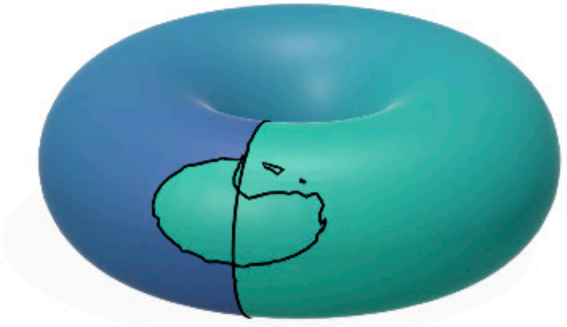
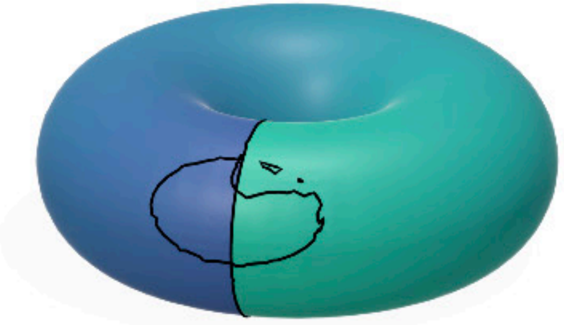


2. Minimize the L^1 norm of jumps across connected components.

v



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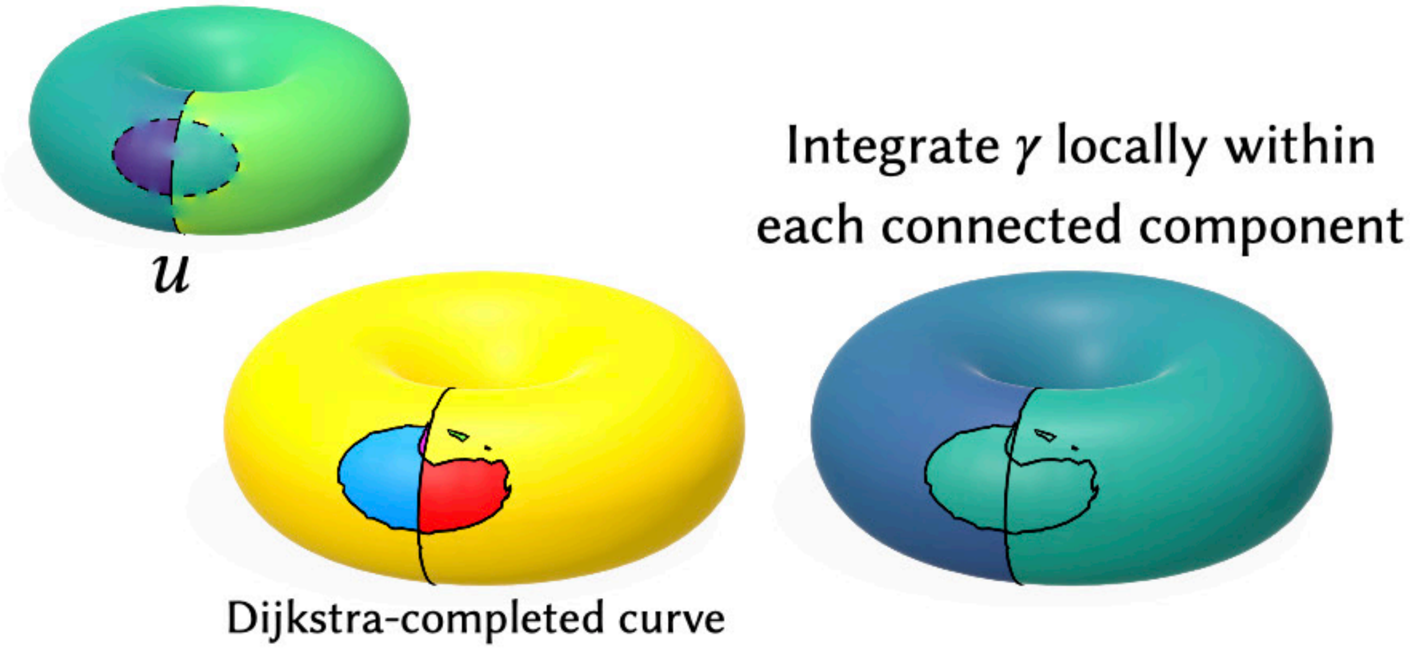

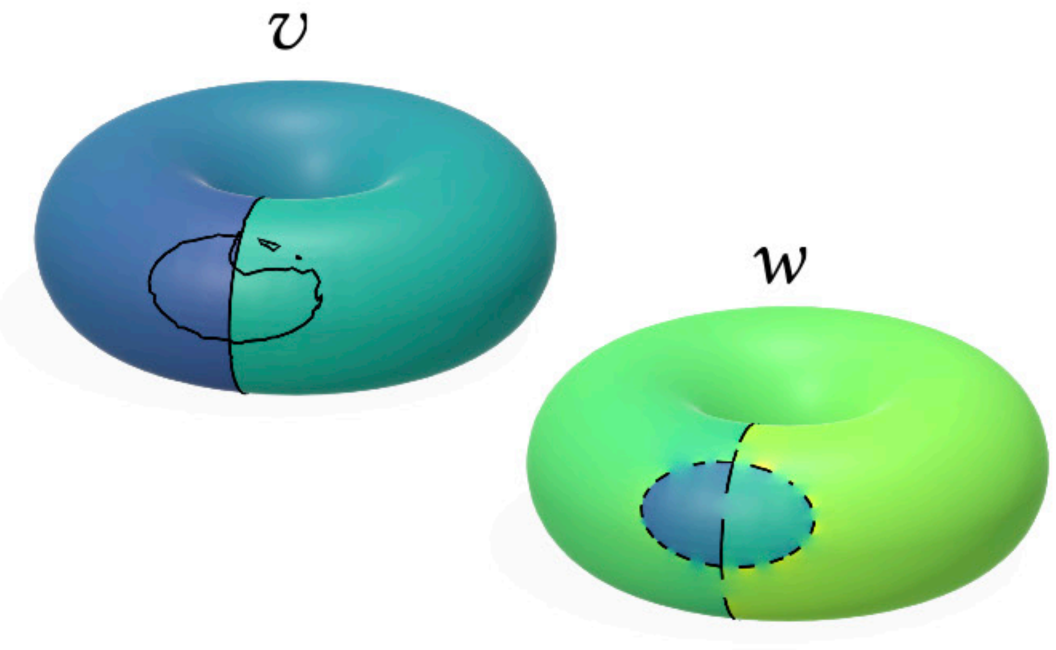

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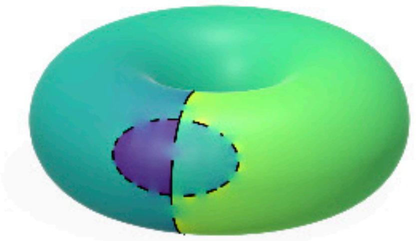

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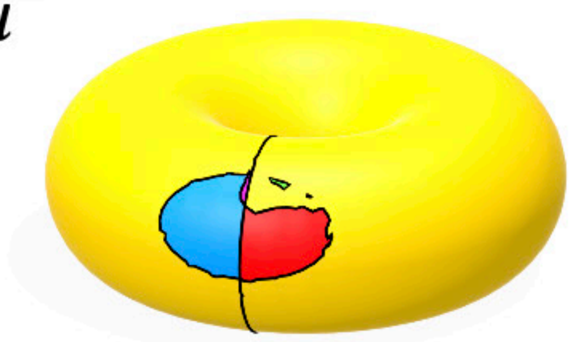

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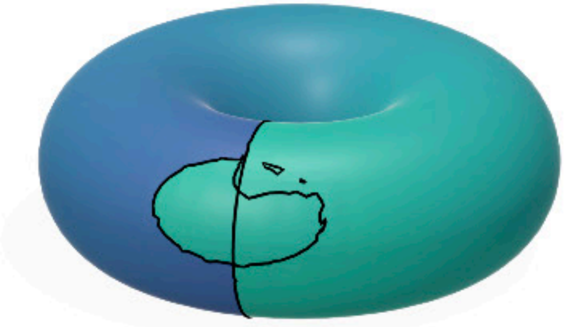

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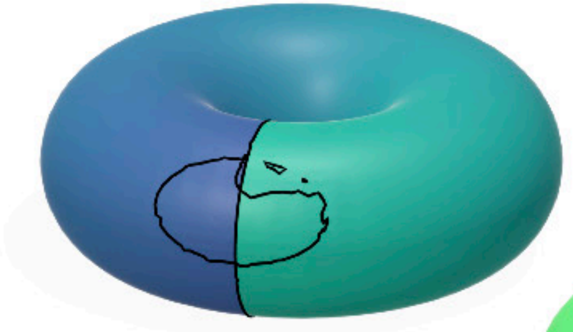
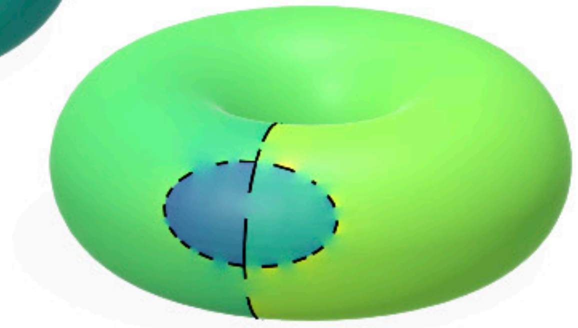
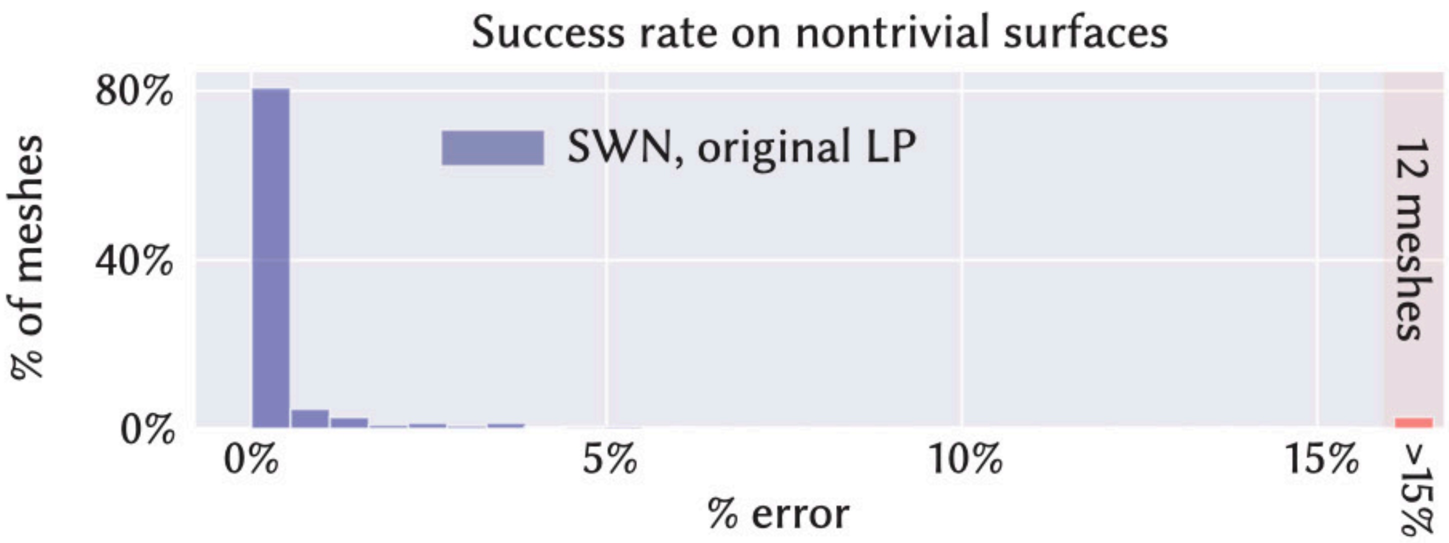


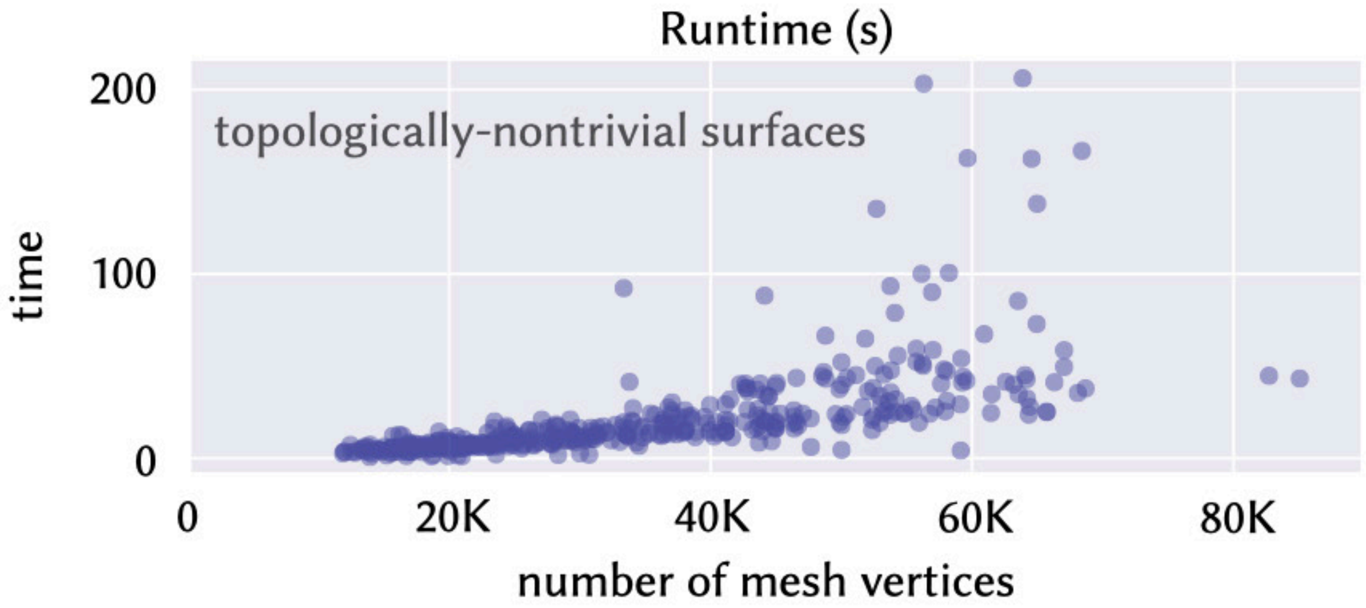
w



Number of DOFs: $|F| \rightarrow$ just a few connected components!

Approximate solutions, 100x faster





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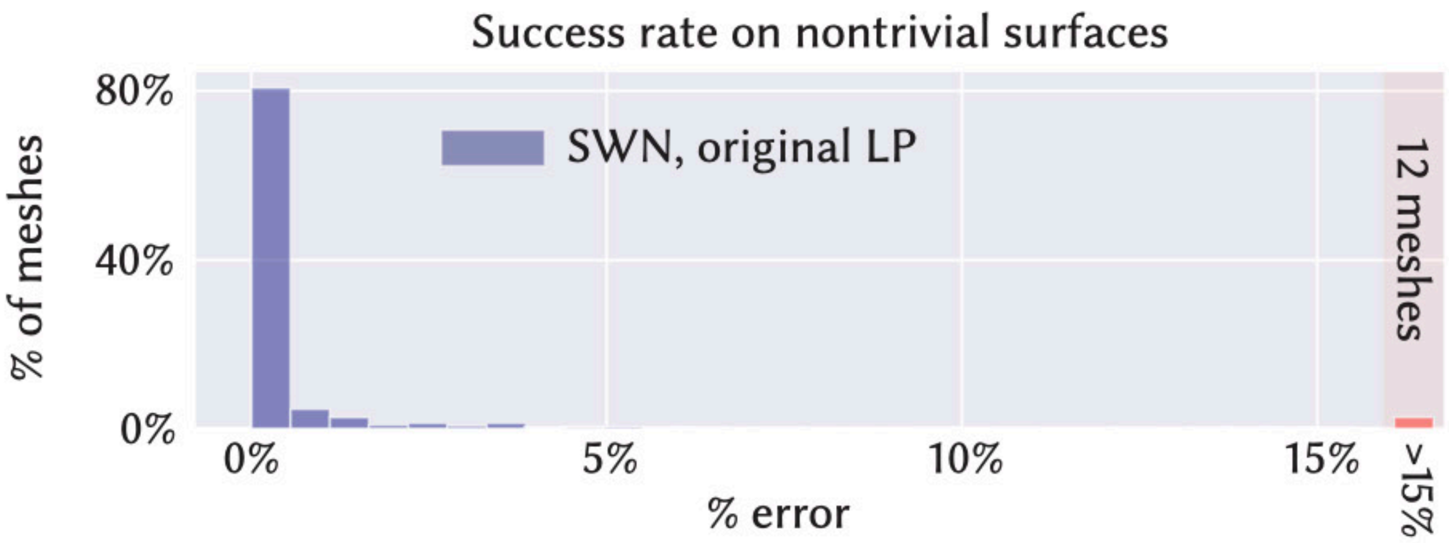

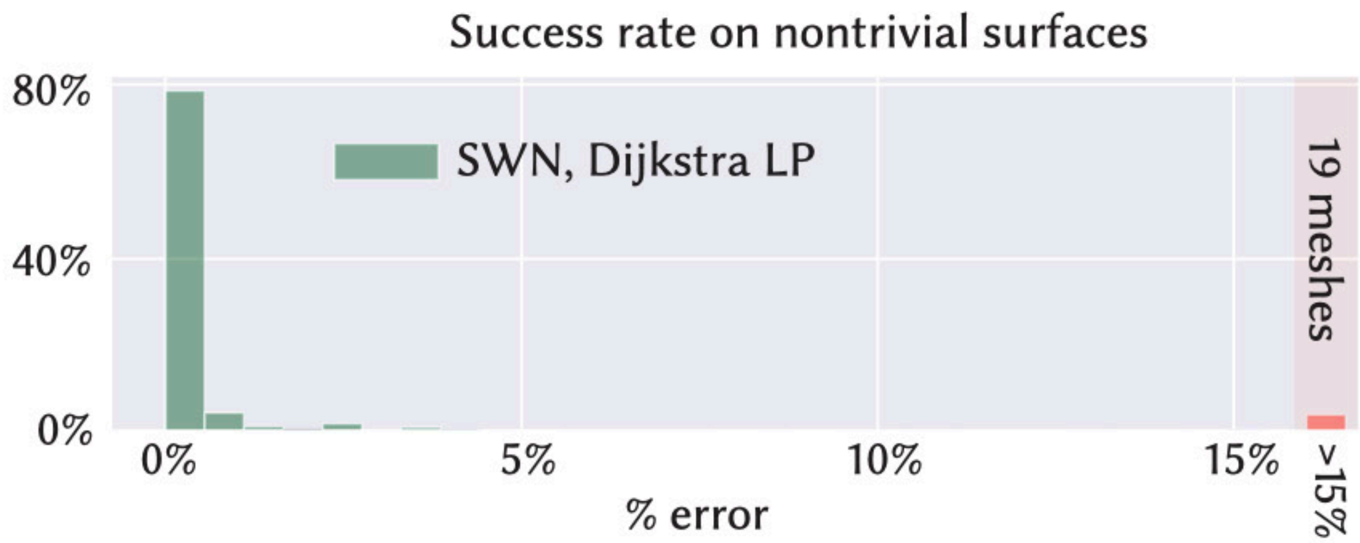

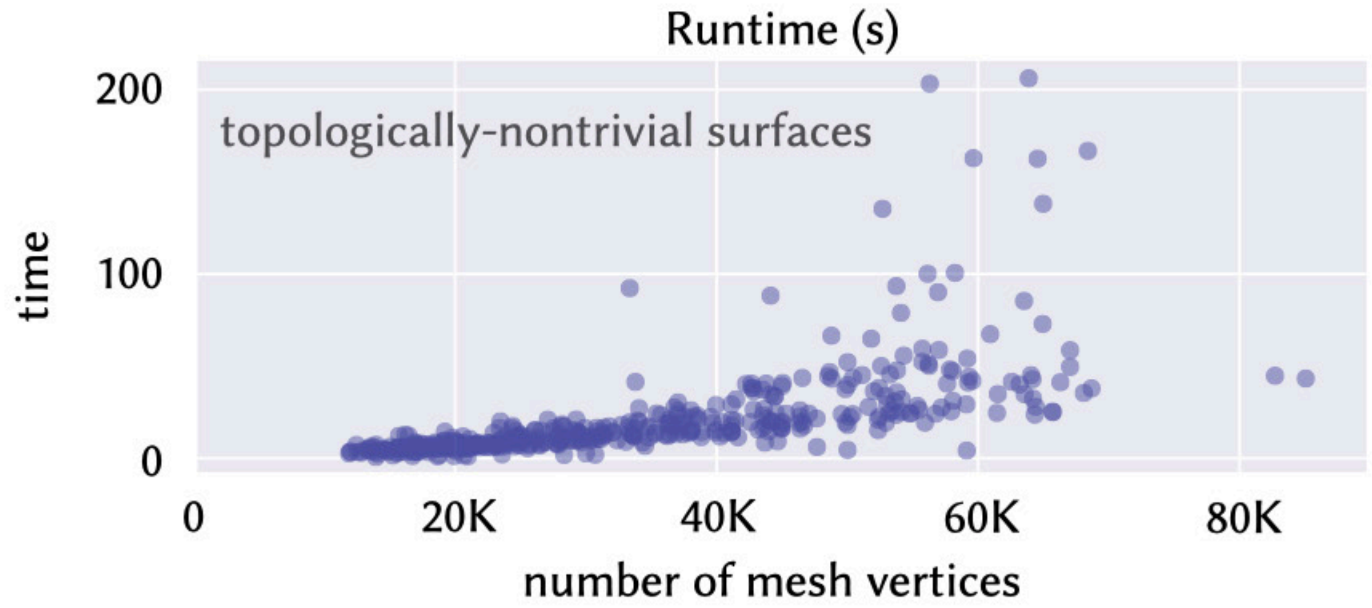

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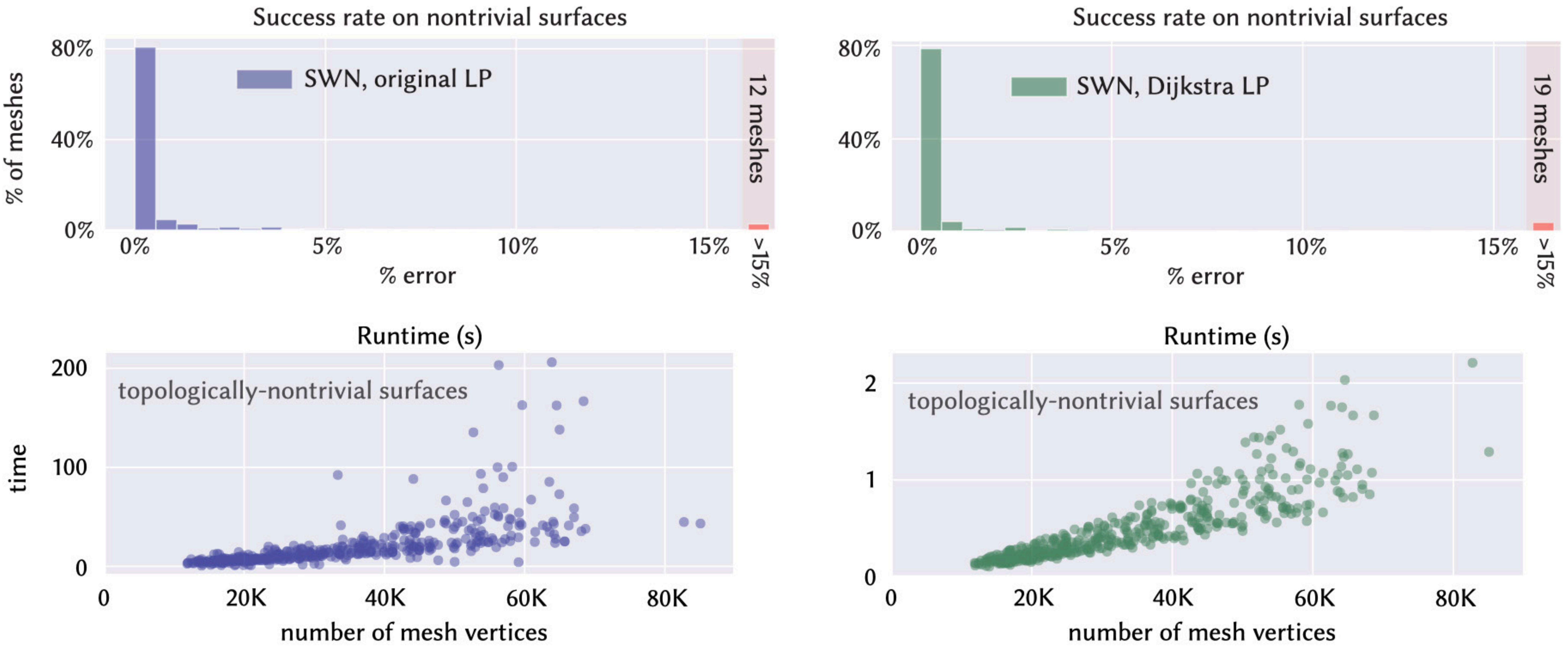

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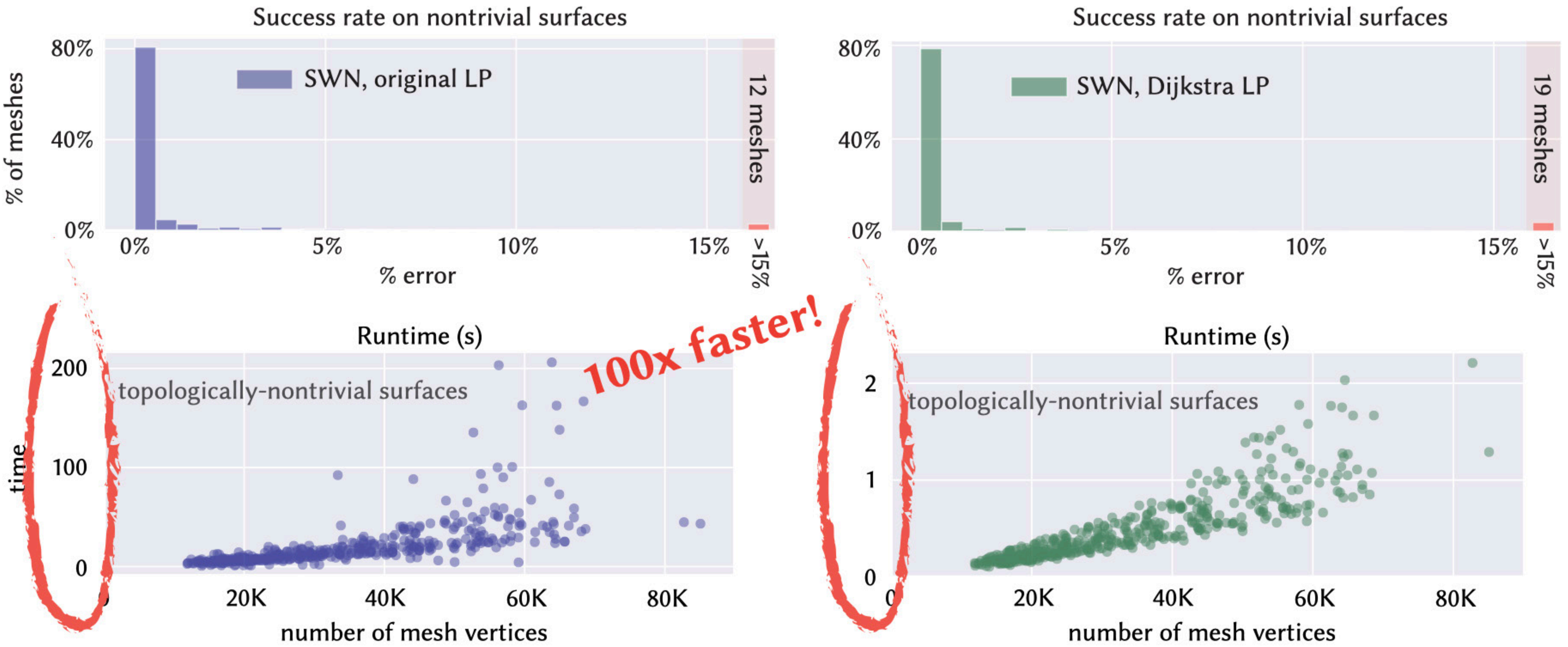

CONCLUSION

Takeaways

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- Cohomology → robust homological geometry processing

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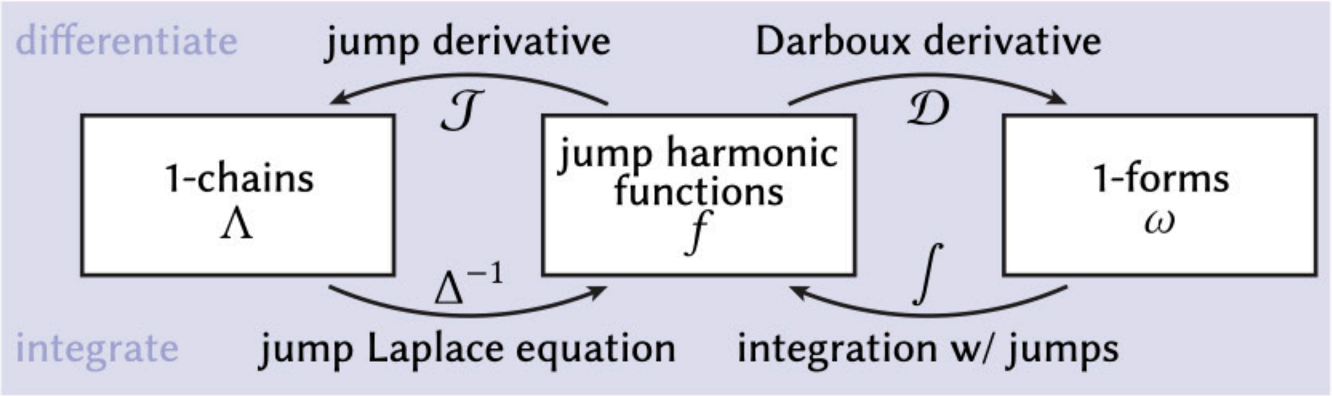
- Cohomology → robust homological geometry processing
  - Duality between curves and 1-forms → use jump harmonic functions to translate between the two



Fun future directions

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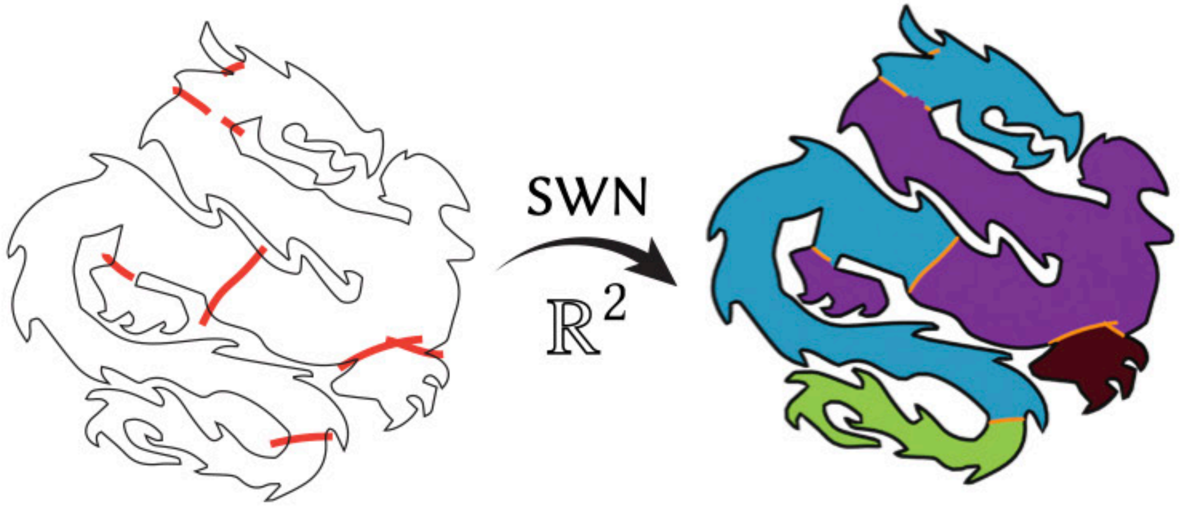
Subsets of $\mathbb{R}^n$:



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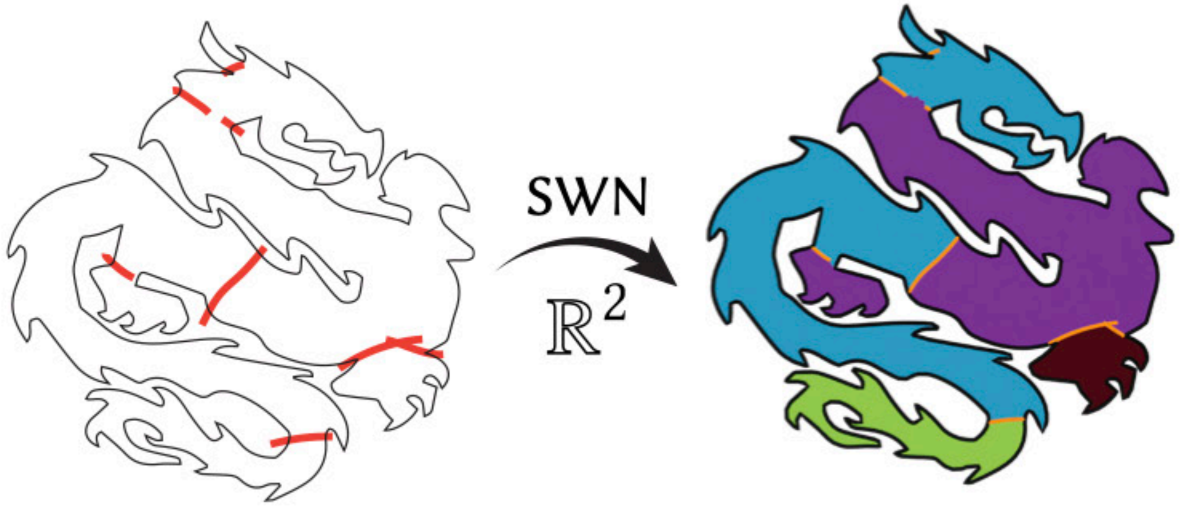

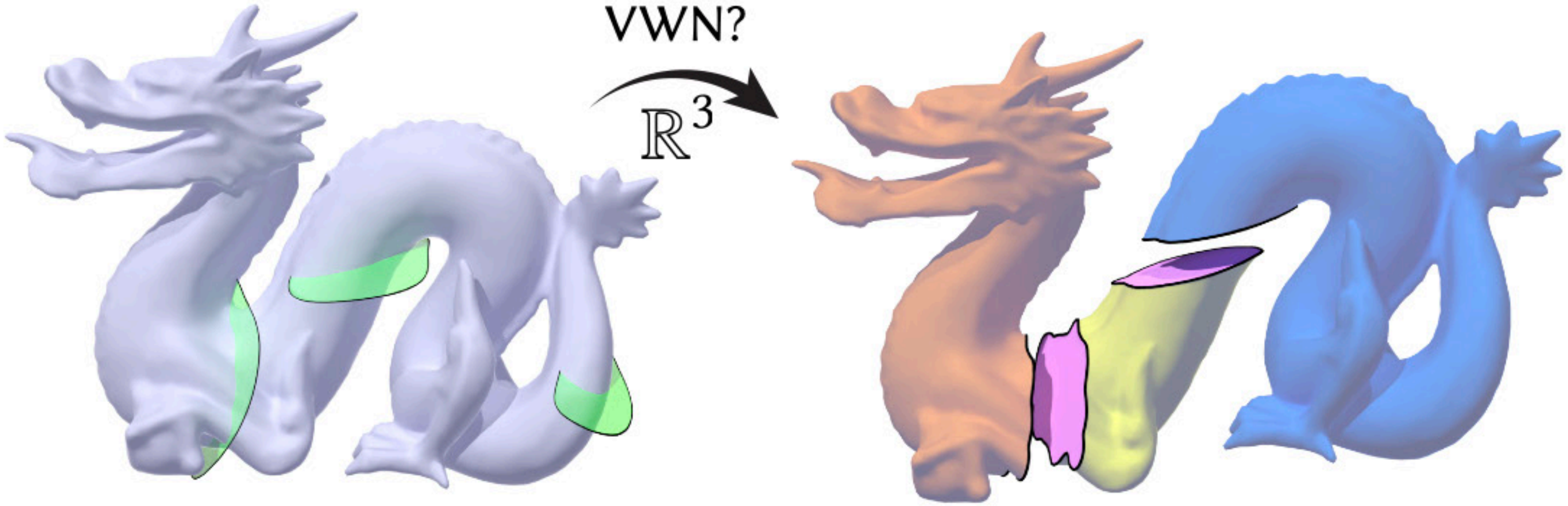

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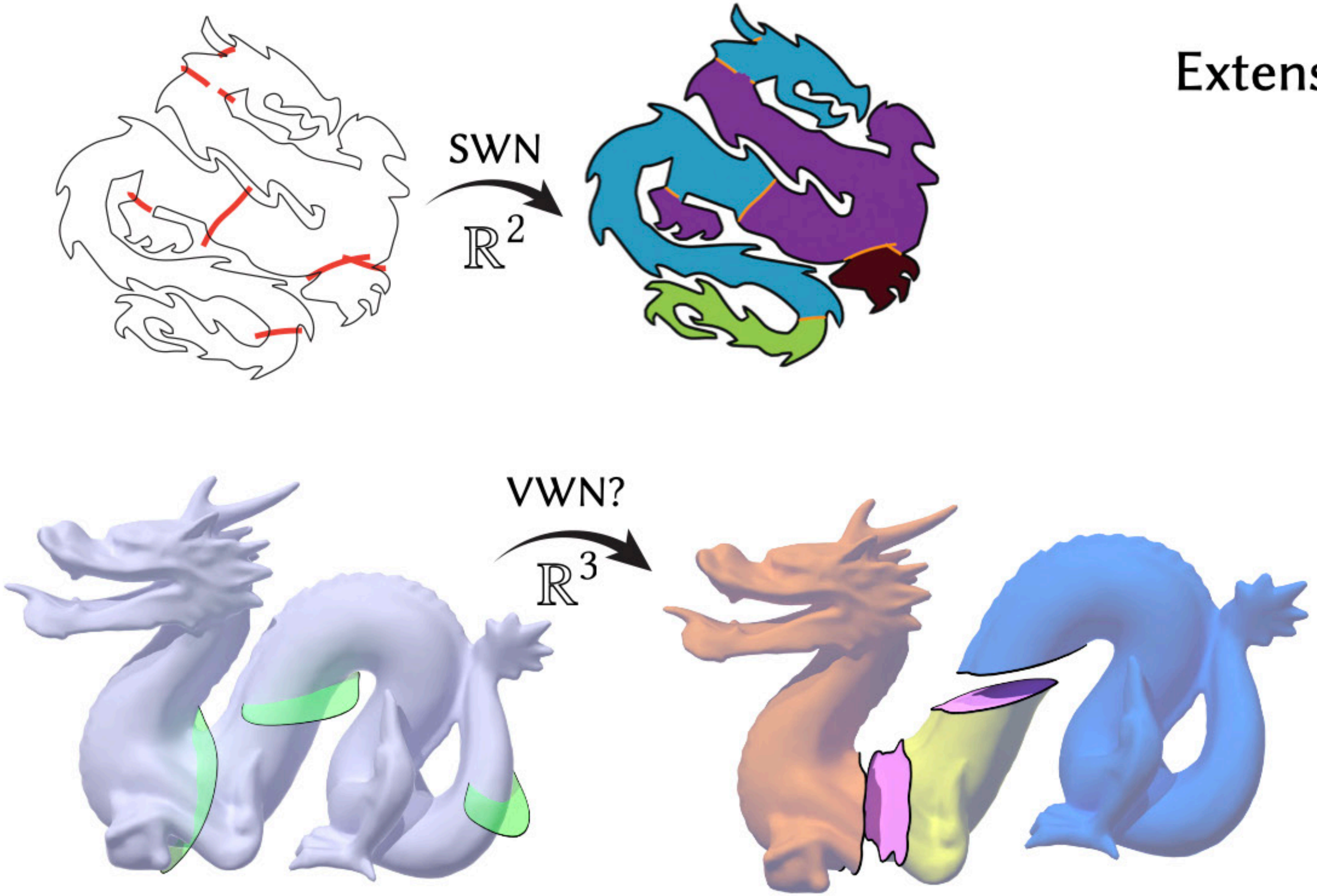
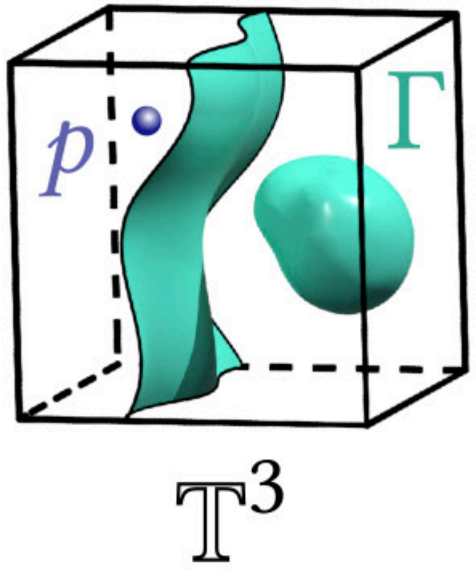
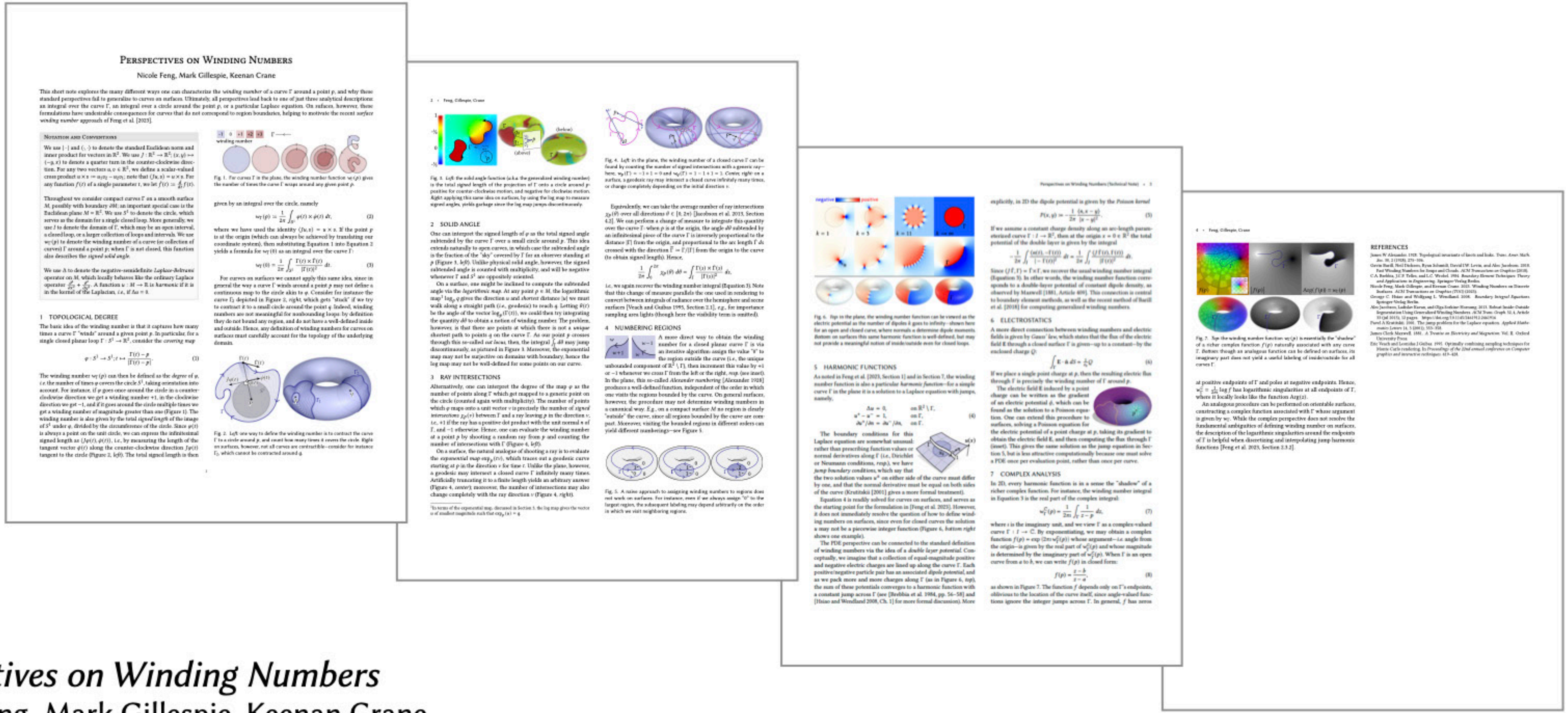


Extension of SWN to higher dimensions,
e.g. periodic domains in 3D.



Winding numbers are everywhere!

Many mathematical & physical interpretations — see our supplemental for details!



Perspectives on Winding Numbers
Nicole Feng, Mark Gillespie, Keenan Crane