Winding Numbers on Discrete Surfaces

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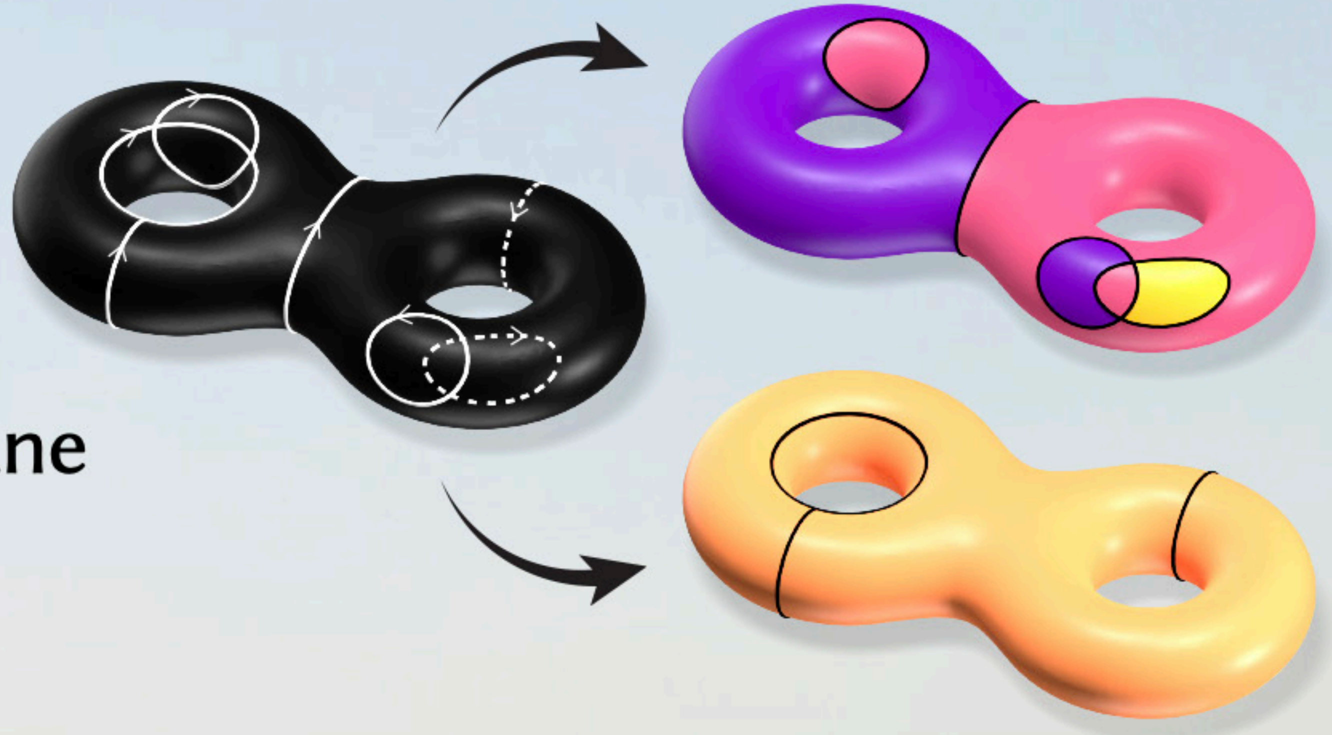
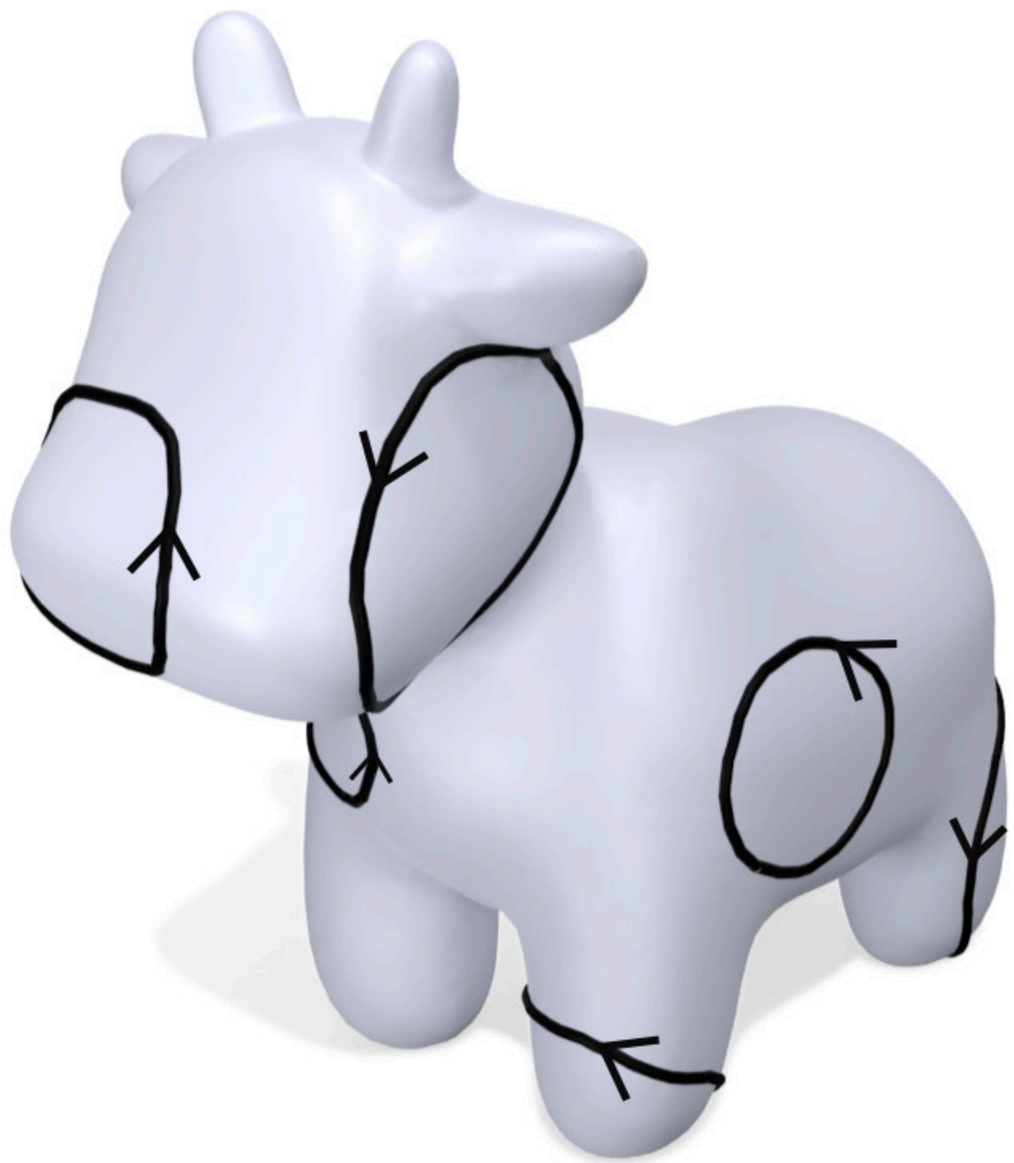
Carnegie Mellon University



SIGGRAPH 2023

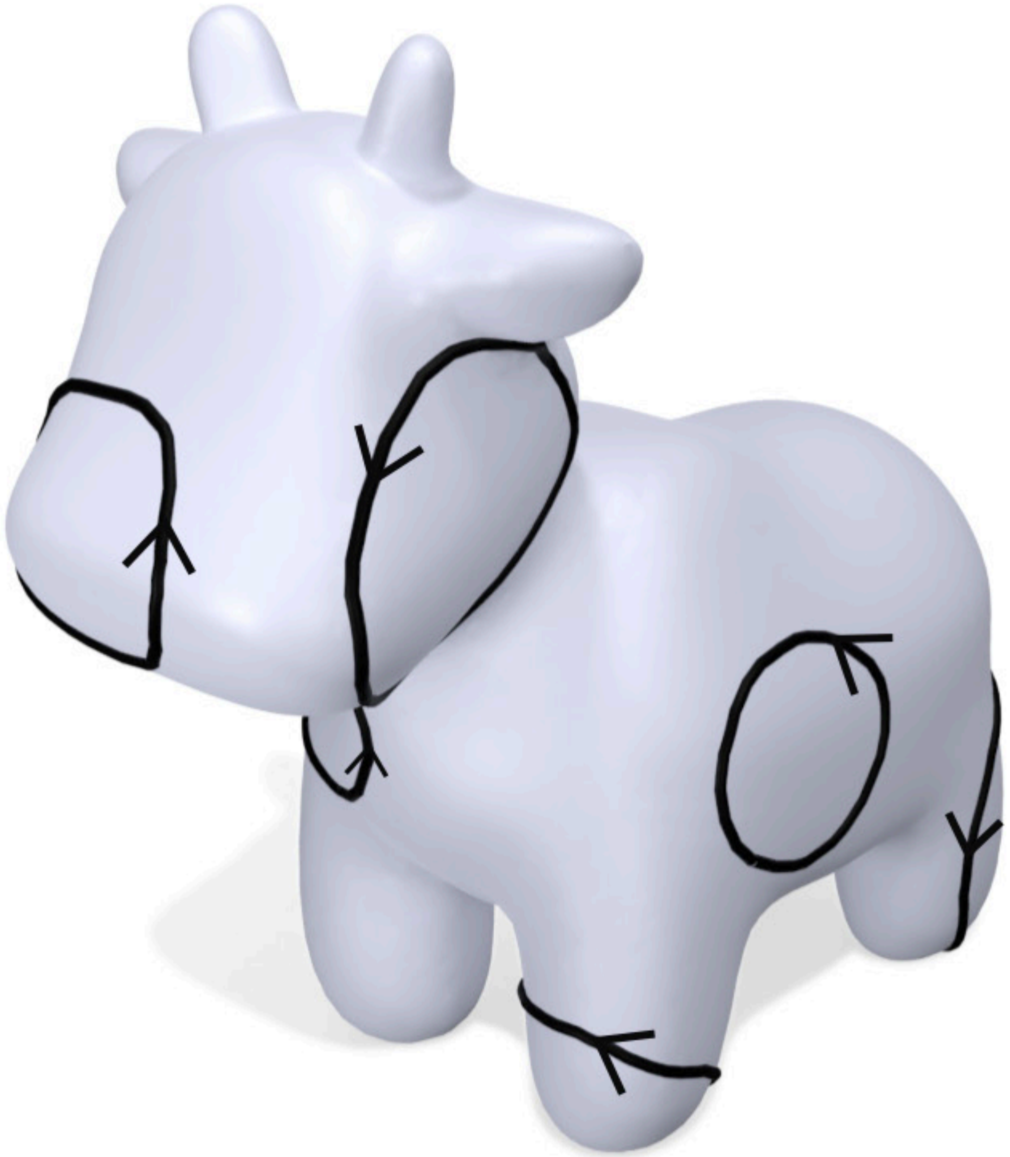
Problem

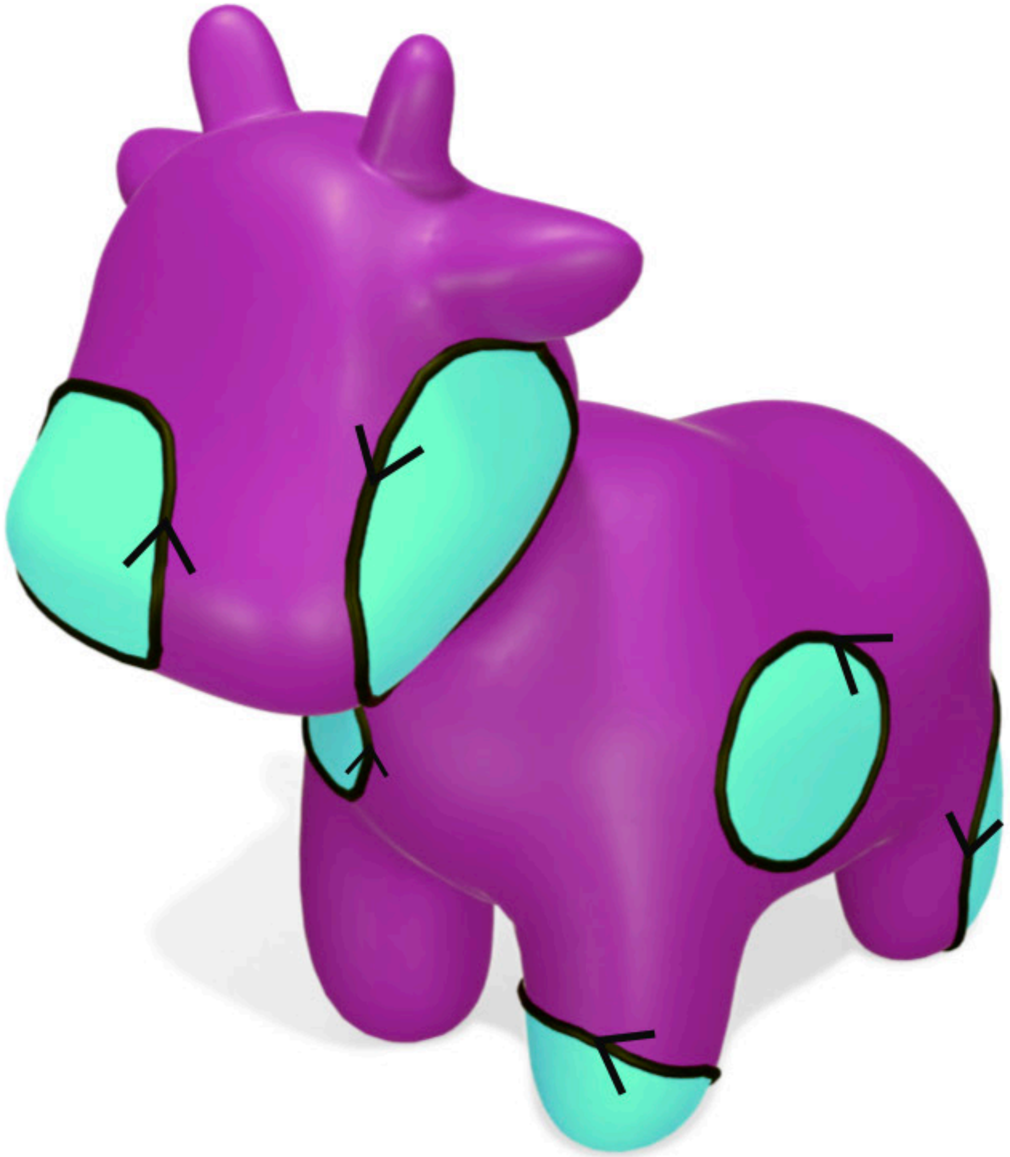
Given curves on a surface...



Problem

Given curves on a surface… …classify points as inside or outside.





Not all closed curves have an inside & outside

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Fence bounds a region:

Not all closed curves have an inside & outside

Fence bounds a region:

Fence doesn't bound a region:

Not all closed curves have an inside & outside

Fence bounds a region:

Bounding

Fence doesn't bound a region:

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Nonbounding

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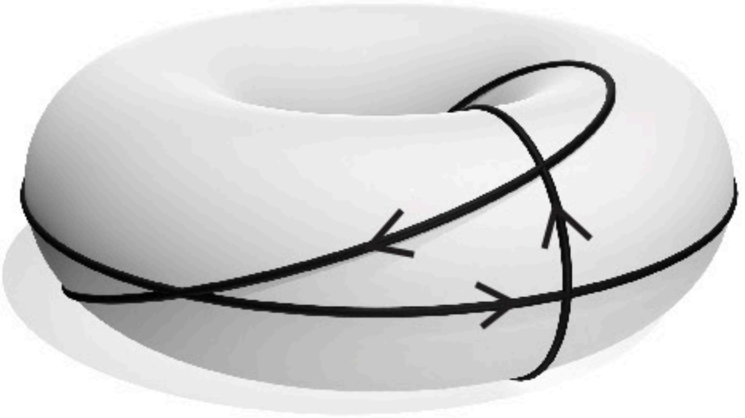
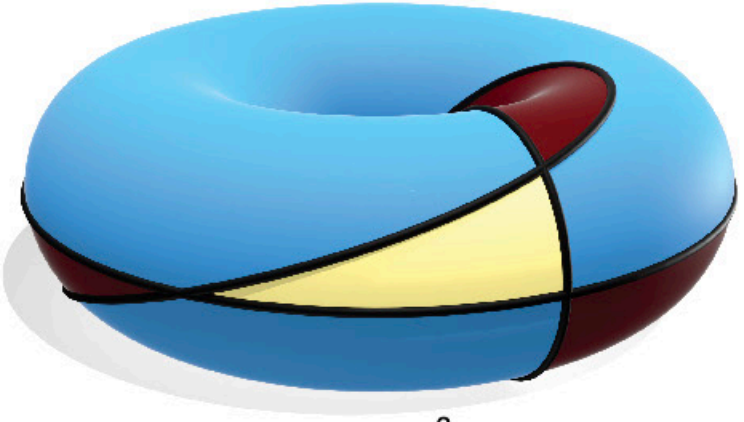
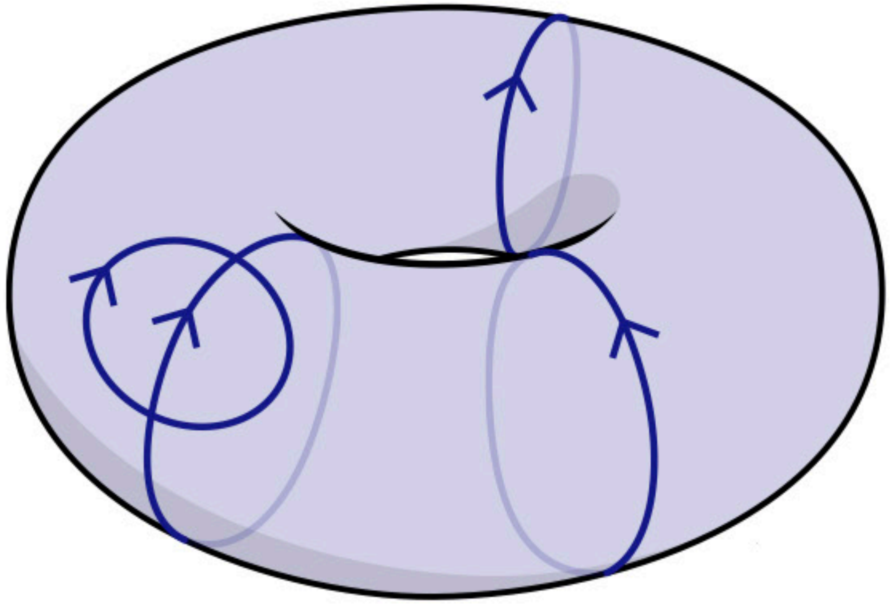




Can't just throw away nonbounding curves

Many ways to partition input into regions

input



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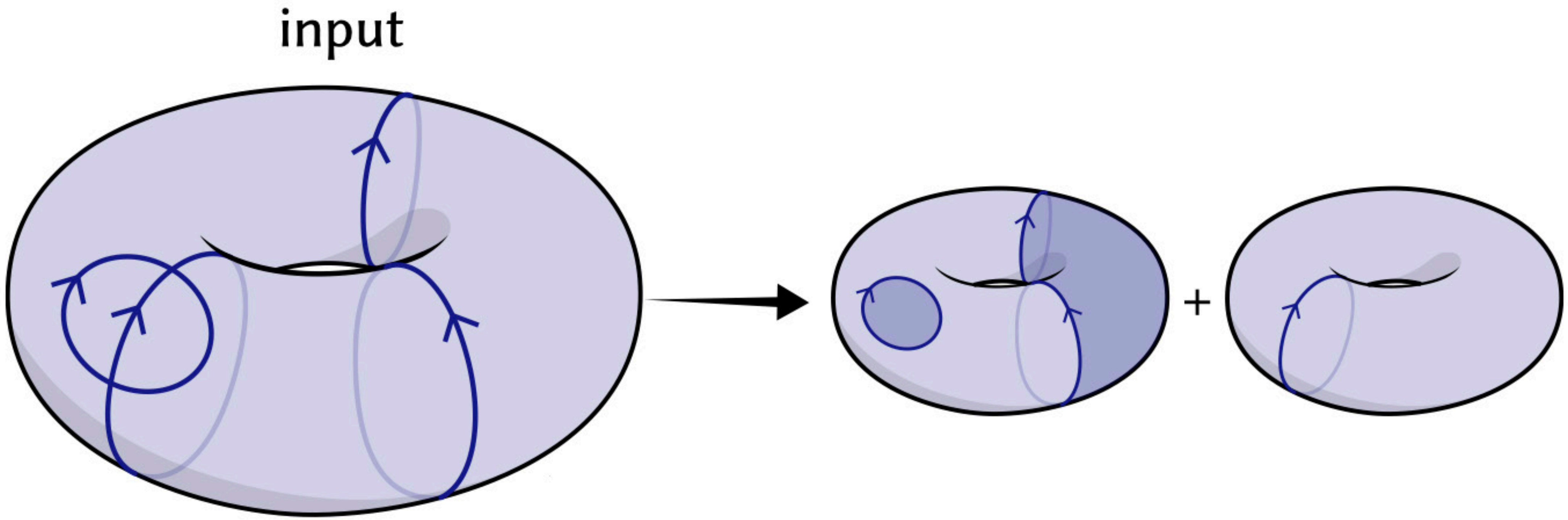

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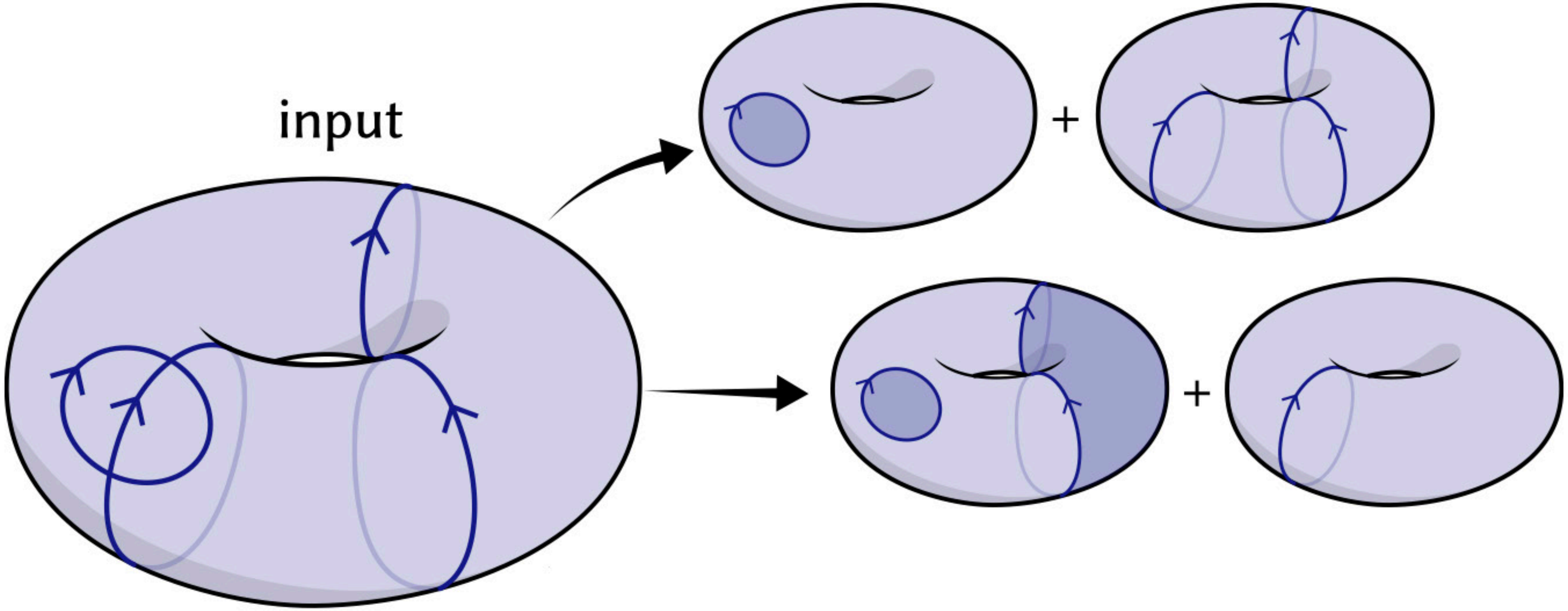

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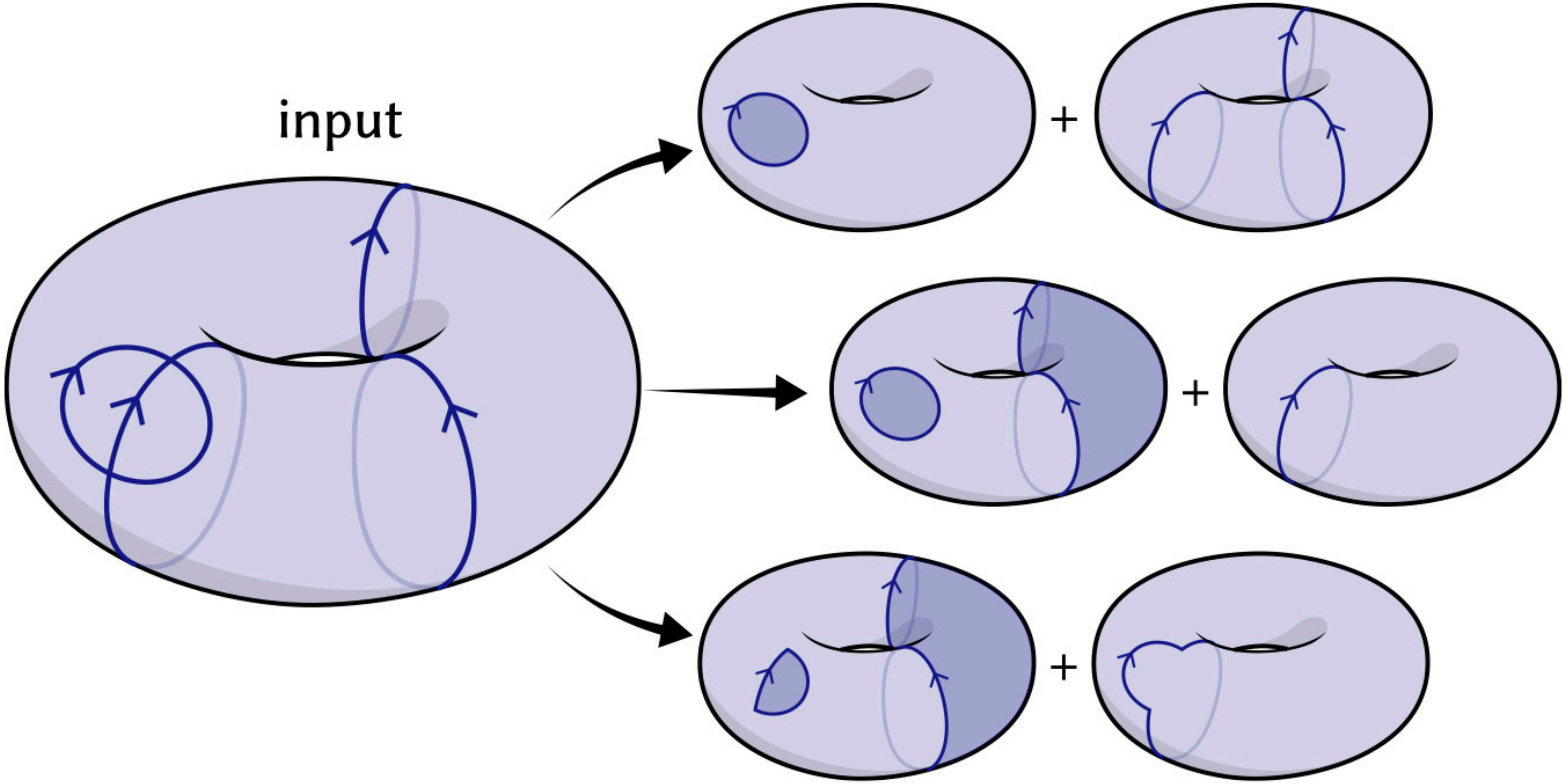
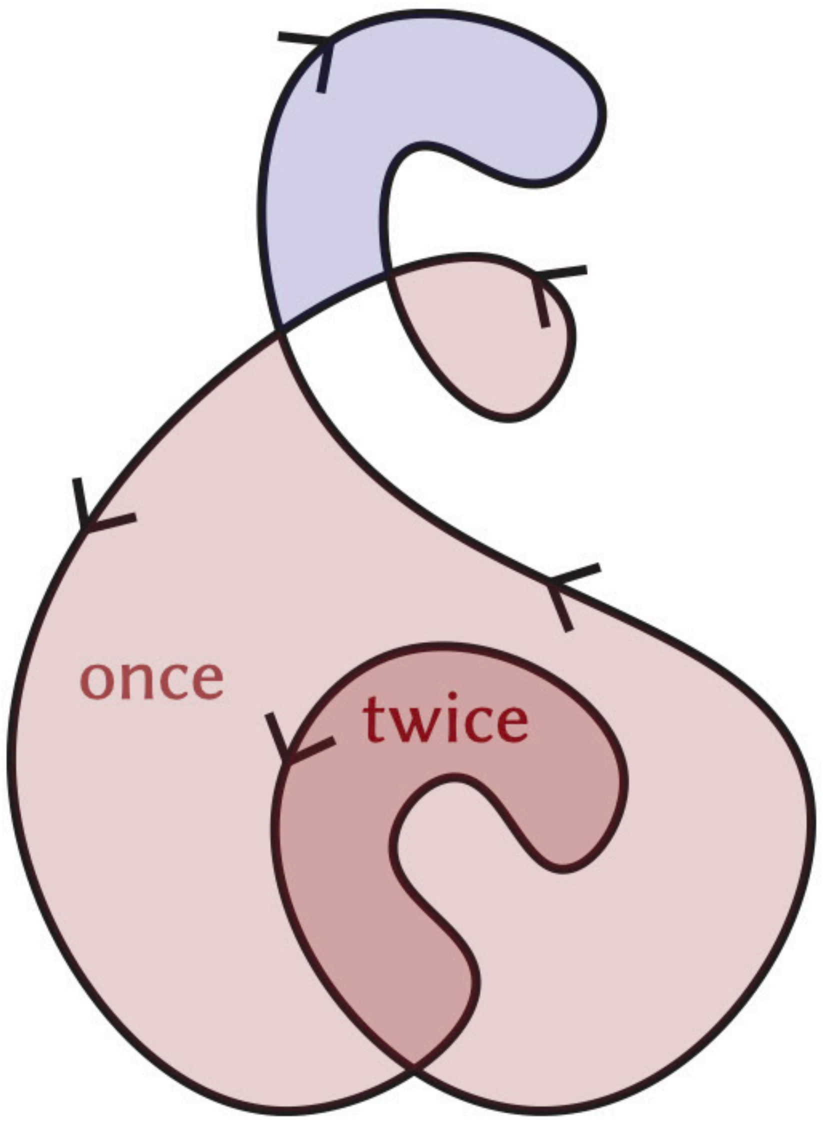


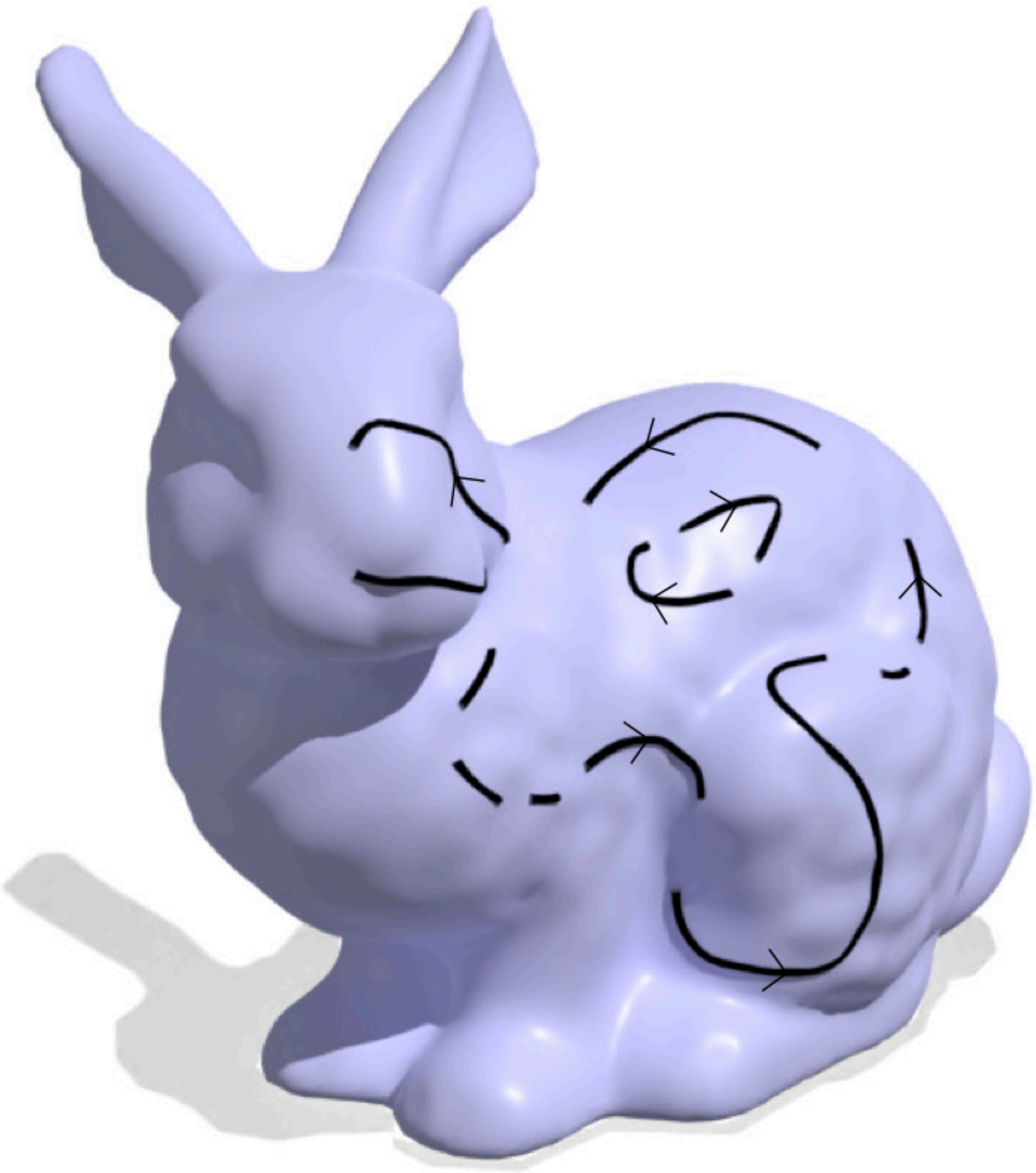
Curves can enclose points multiple times

-1	0	+1	+2

winding number



Inside/outside of broken curves?



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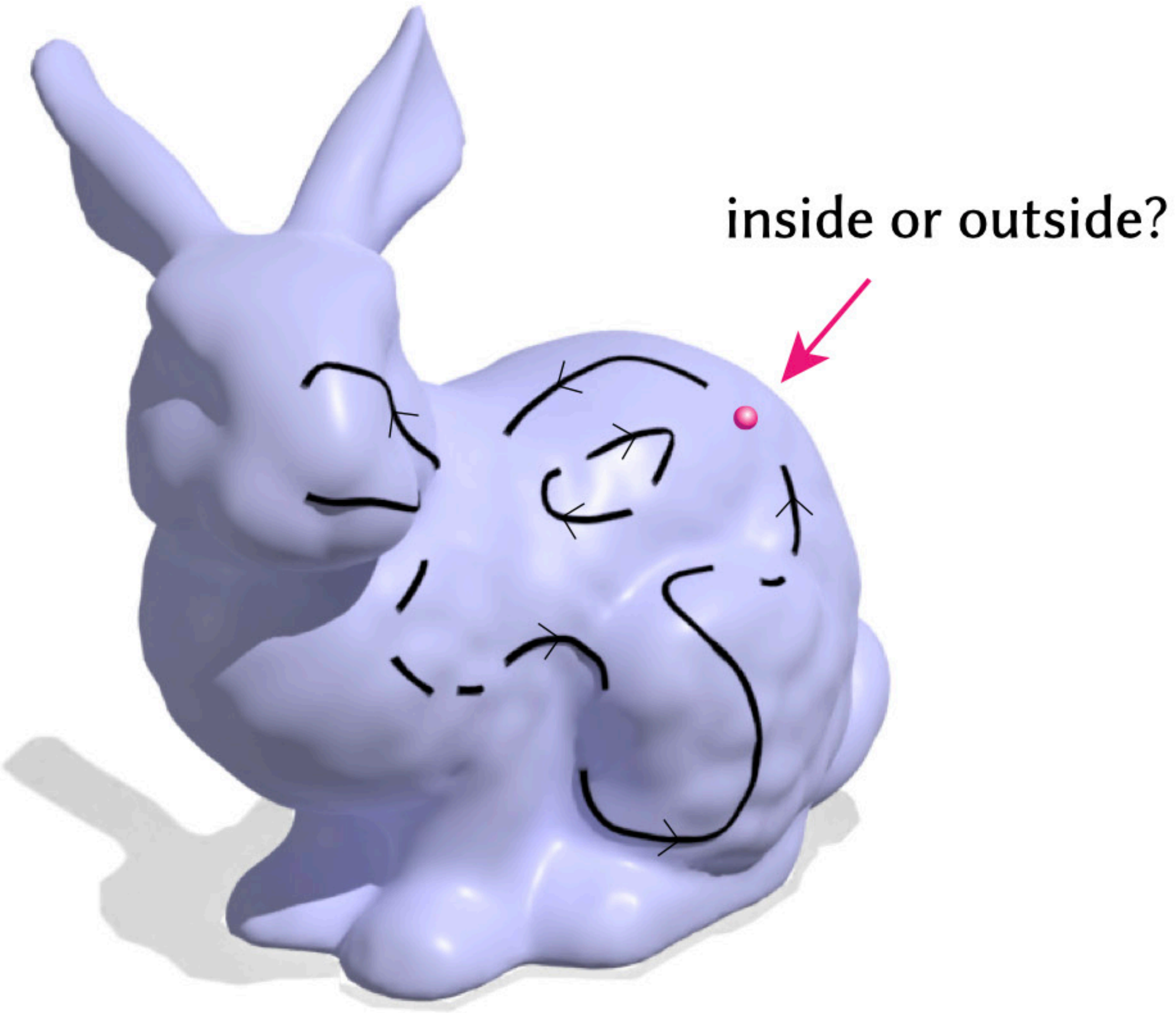

Surface Winding Numbers (SWN)

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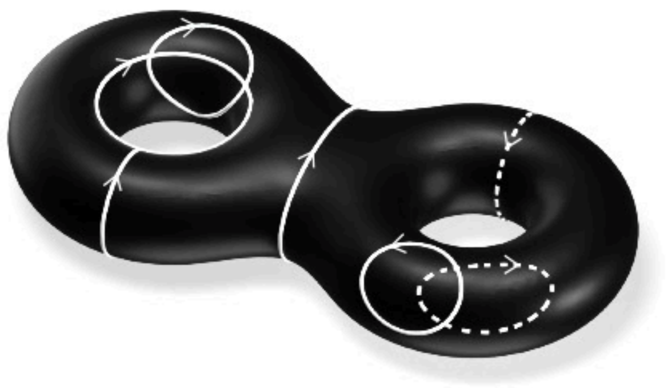
Input: subset of edges in a triangle mesh



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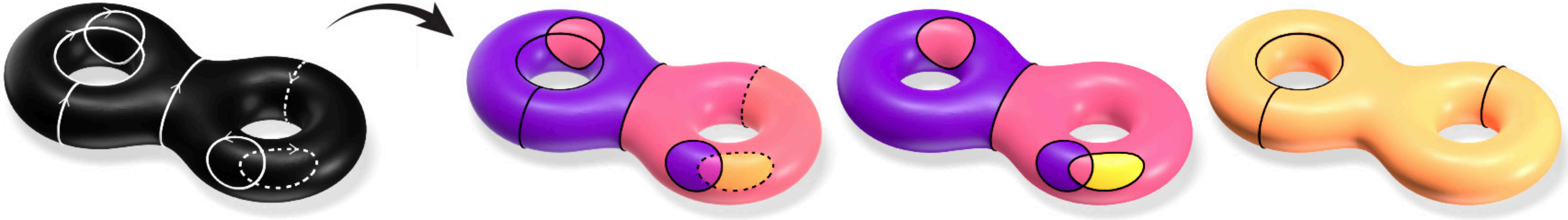

Outputs: winding number function | bounding components | nonbounding components

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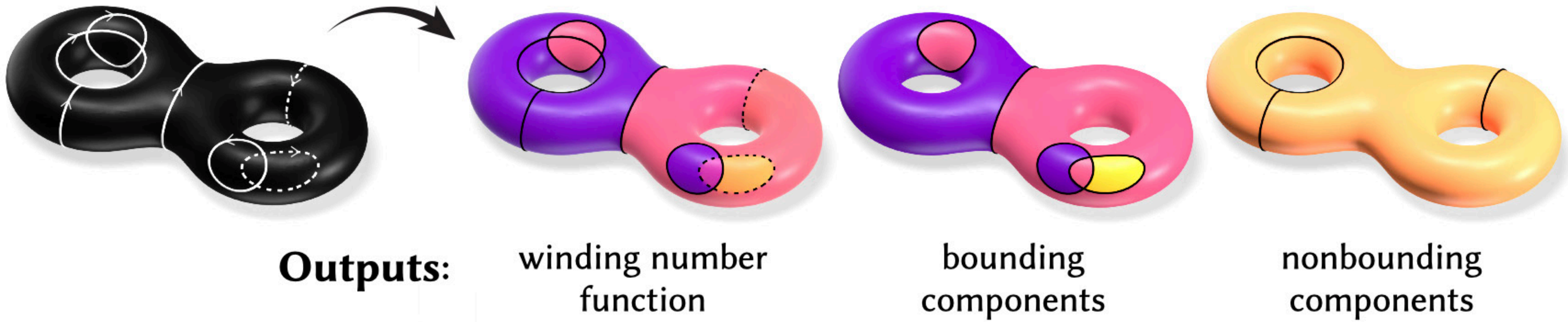
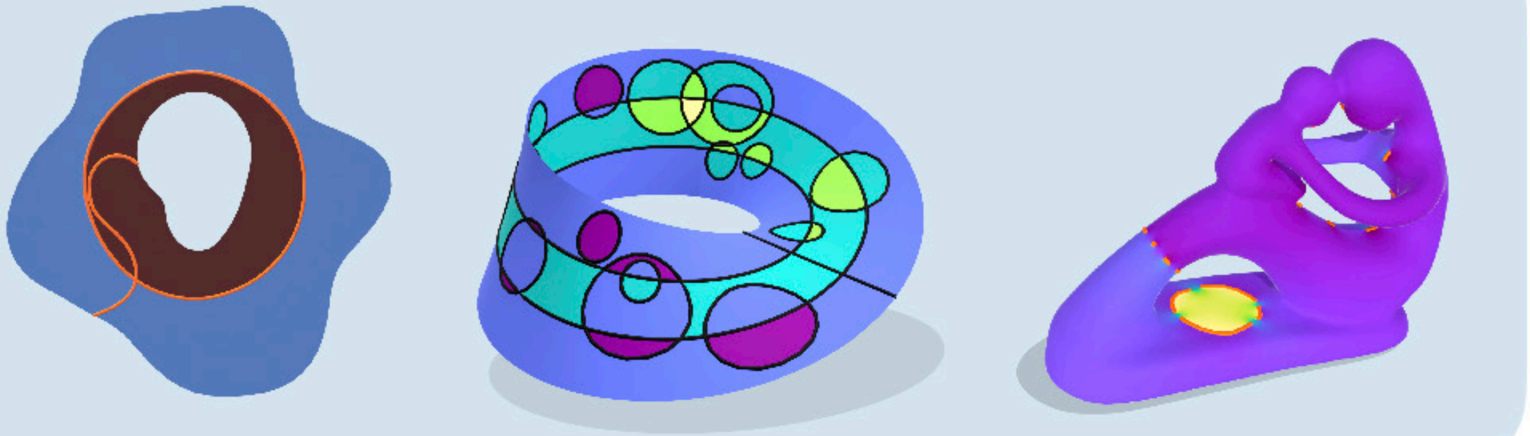


Handles general topology, broken curves



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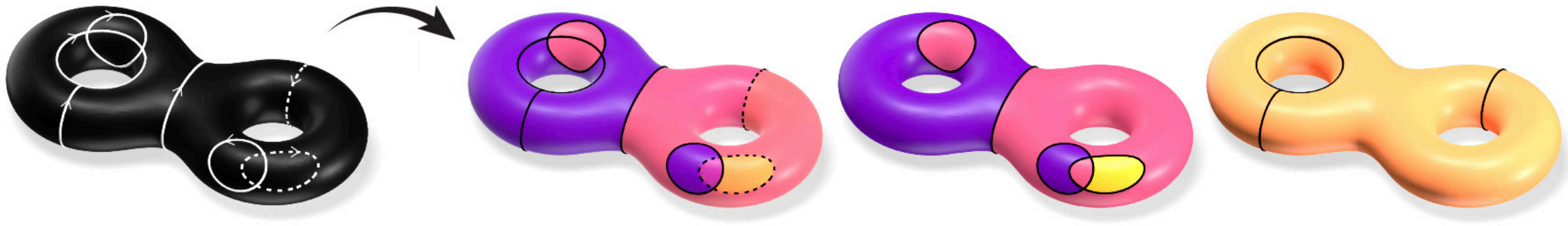

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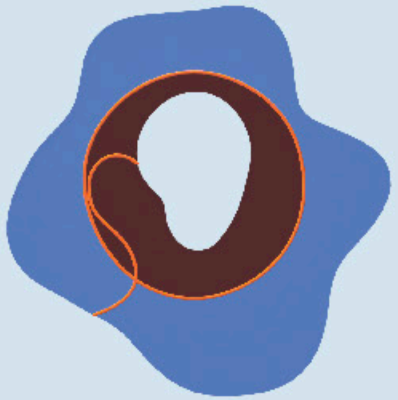
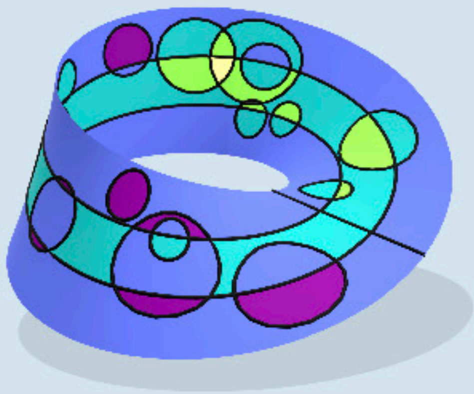
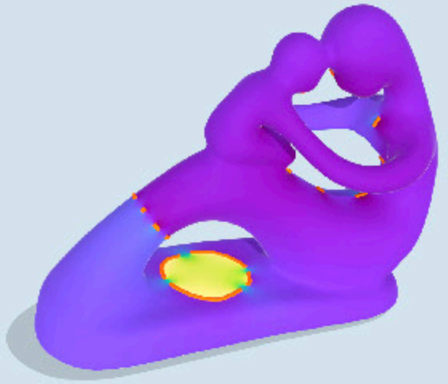
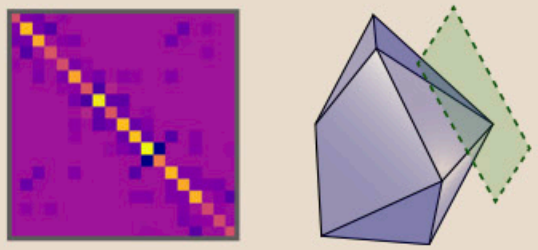






sparse Poisson problems + LP



Winding numbers are useful for geometry processing!

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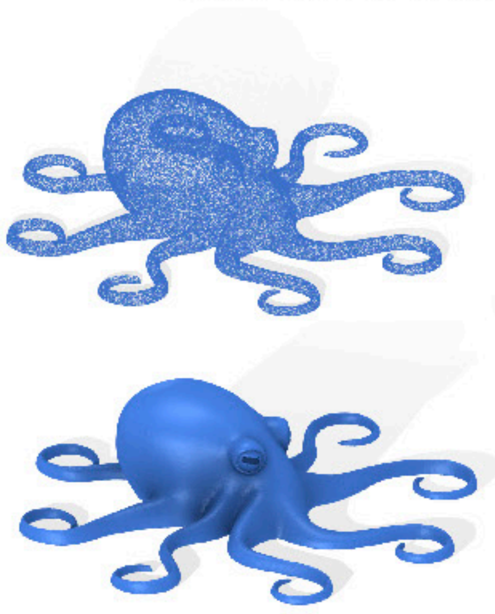
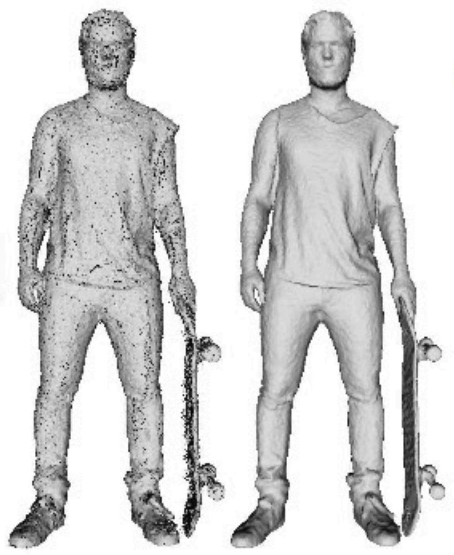
surface reconstruction

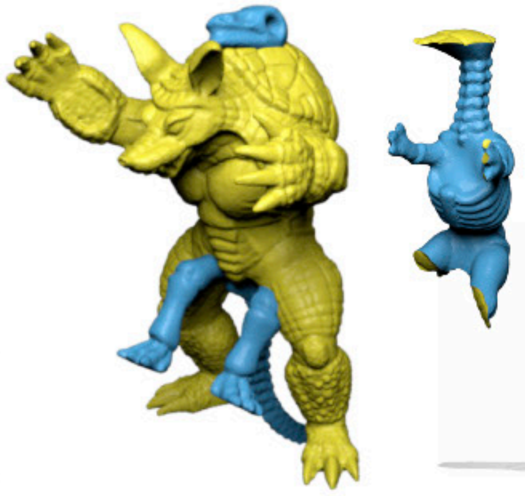


[Barill et al. 2018]

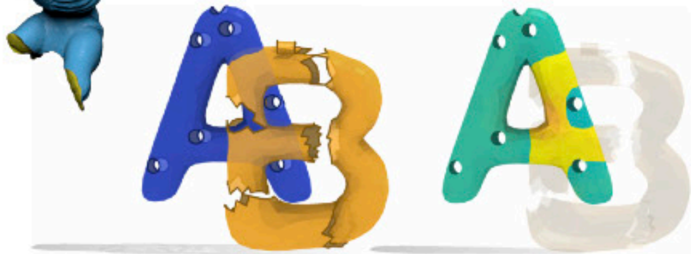


[Collet et al. 2015]

mesh booleans



[Zhou et al. 2016]



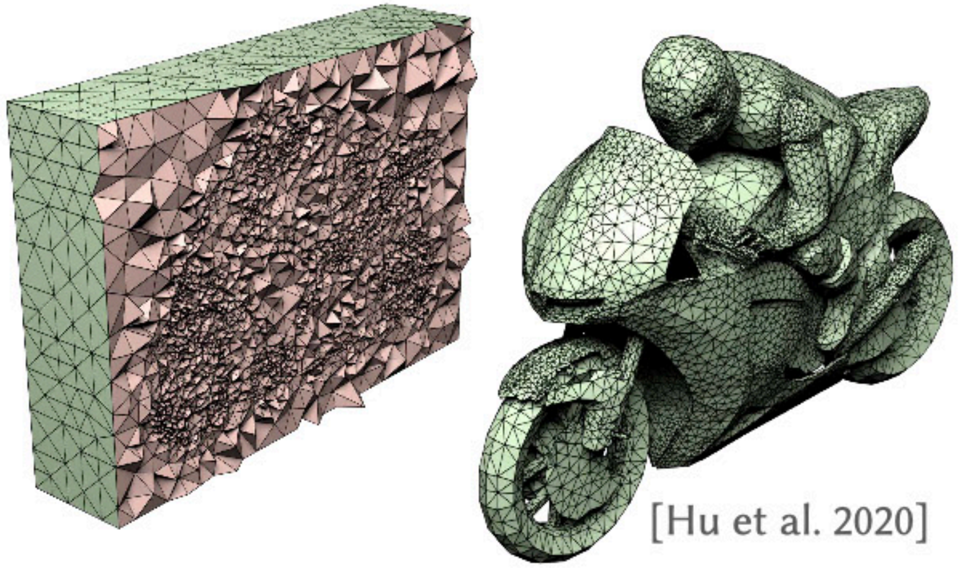
[Barill et al. 2018]

iterative normal estimation

[Xu et al. 2023]

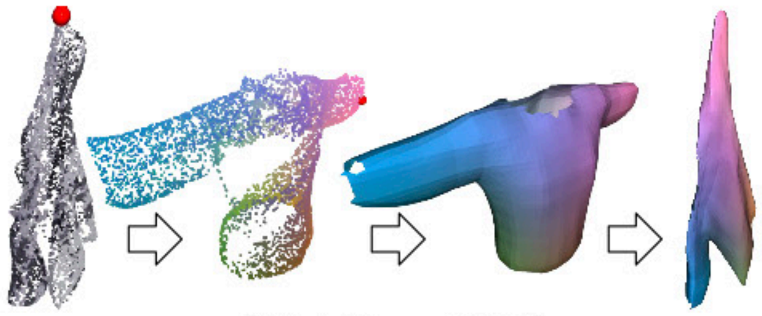
[Hou et al. 2022]

meshing

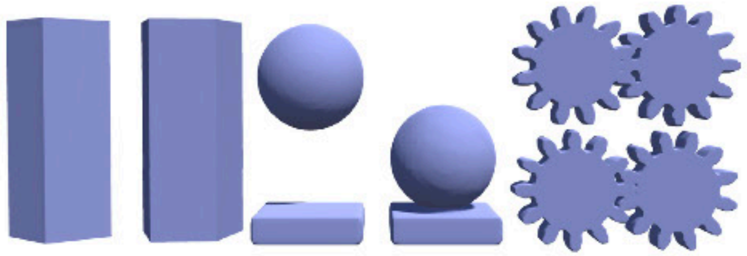


[Hu et al. 2020]

geometric preprocessing



[Chi, Song 2021]



[Dvořák et al. 2021]

History

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(Generalized) winding number = solid angle

[Euler 1781; Lagrange 1798; Gauss 1838, Maxwell 1881...]

On Solid Angles.

417.] We have already proved that at any point P the potential due to a magnetic shell is equal to the solid angle subtended by the edge of the shell multiplied by the strength

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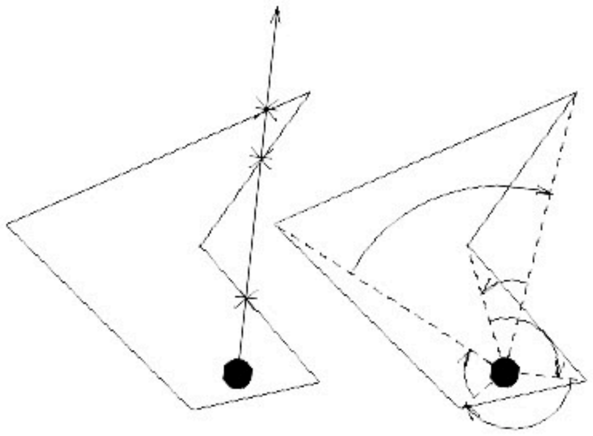
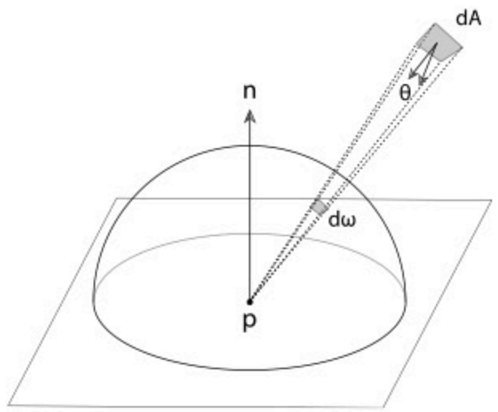
Winding number & solid angle in graphics

[Shimrat 1962; Haines 1994; Goral et al. 1984; Veach & Guibas 1995...]



[Haines 1994]



[Pharr et al. 2018]

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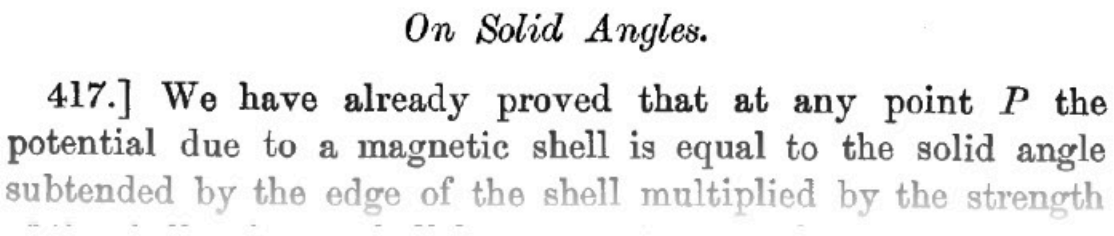

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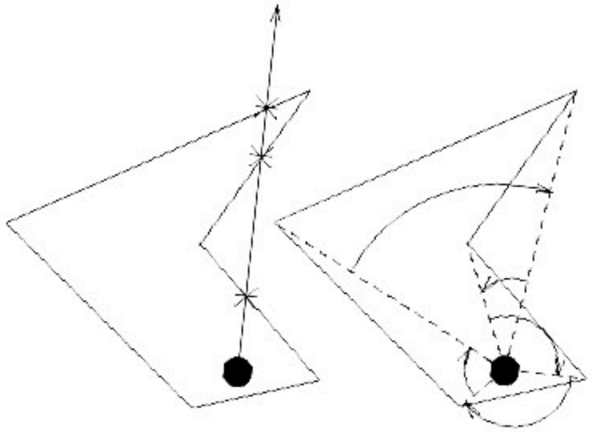

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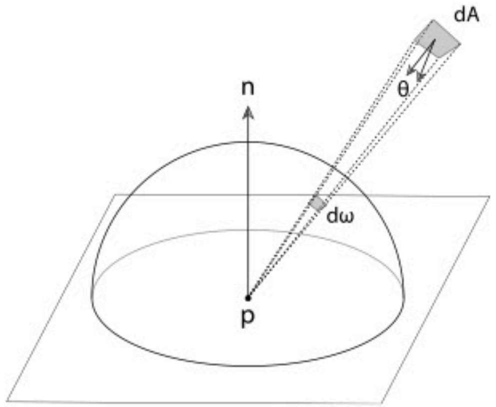
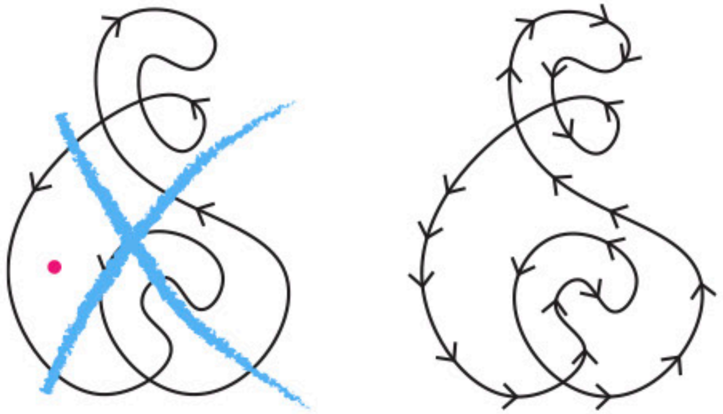


[Pharr et al. 2018]

~~Winding~~ Turning number on surfaces

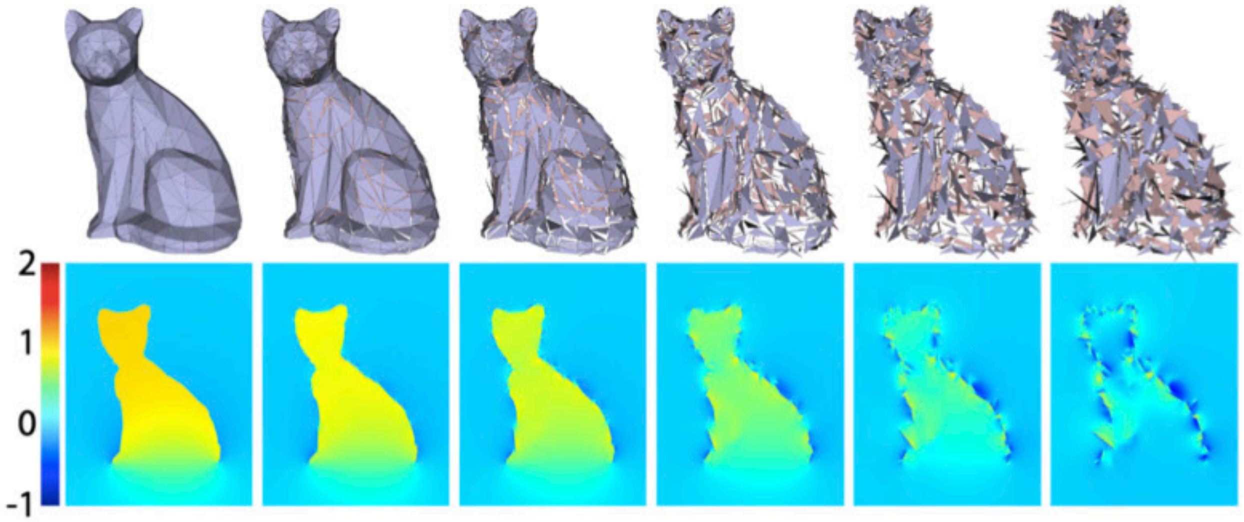
[Reinhart 1960, 1963; Chillingworth 1972; Humphries & Johnson 1989; McIntyre & Cairns 1993; Chernov & Rudyak 2009]



Related work

Poisson Surface Reconstruction = Generalized Winding Number

[Kazhdan et al. 2006] [Jacobson et al. 2013]



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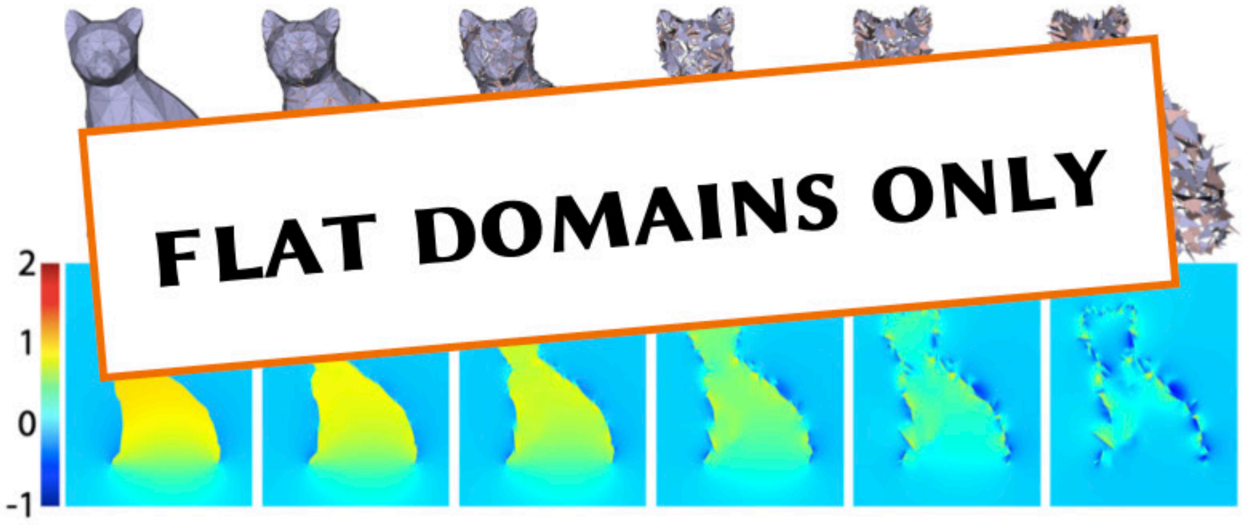

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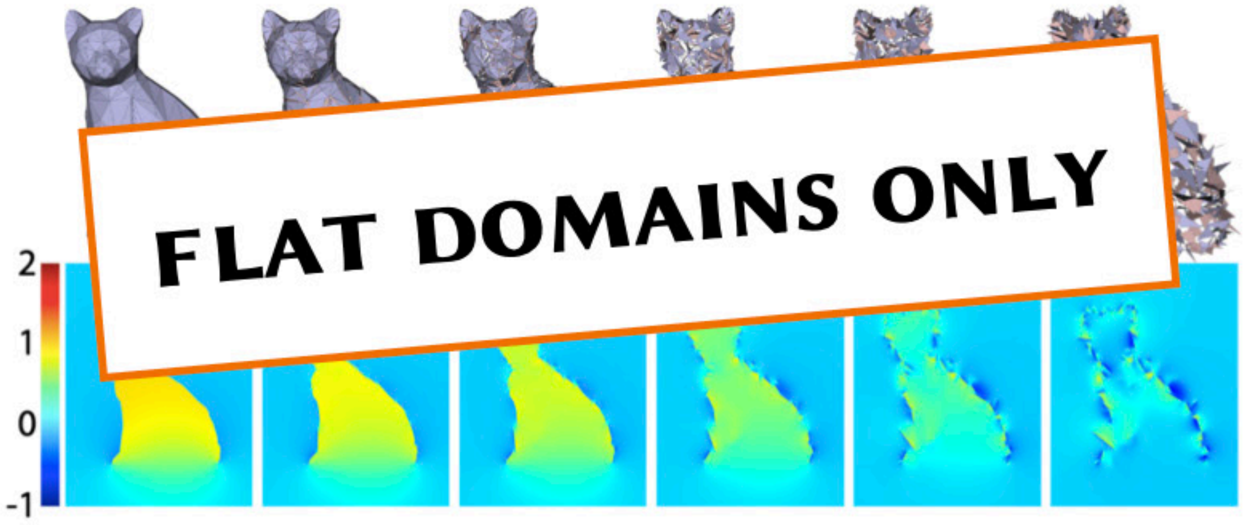

[Jacobson et al. 2013]

BoolSurf
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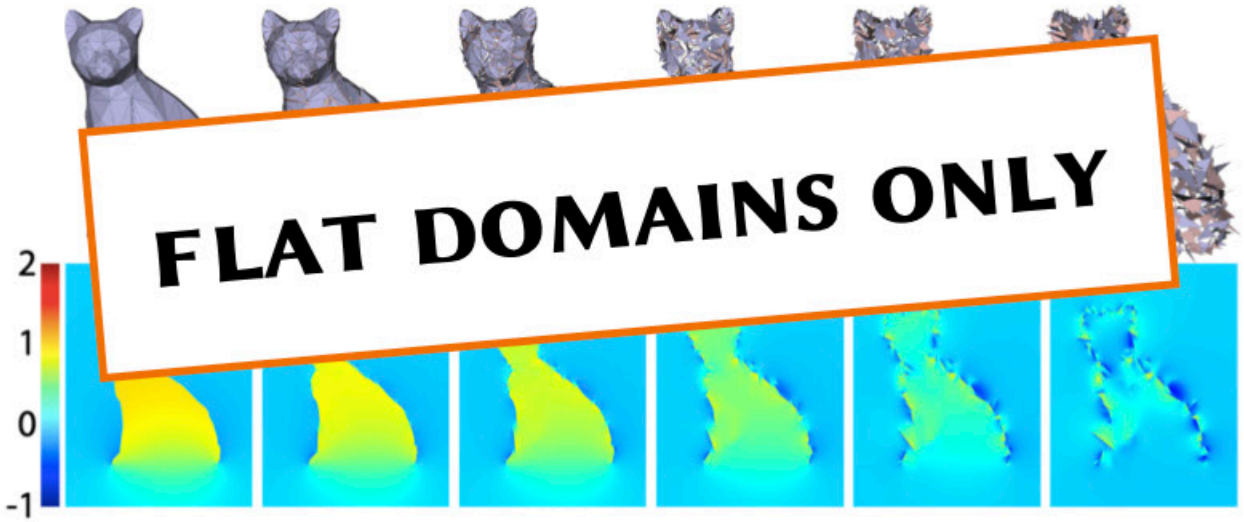

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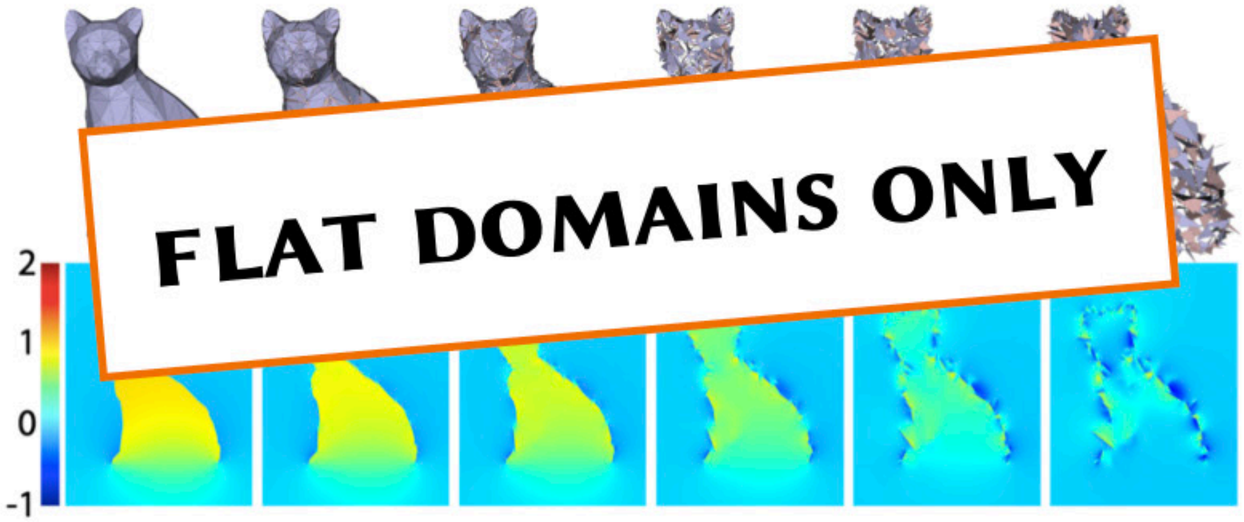


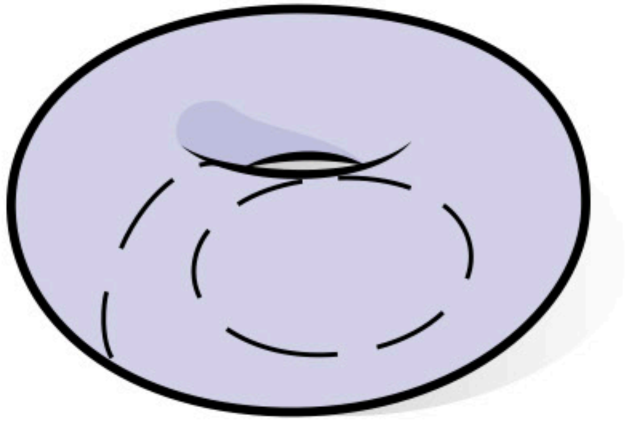
[Jacobson et al. 2013]

We include the best of both worlds (and more)

ALGORITHM

Basic idea: treat curves as vector fields

curves



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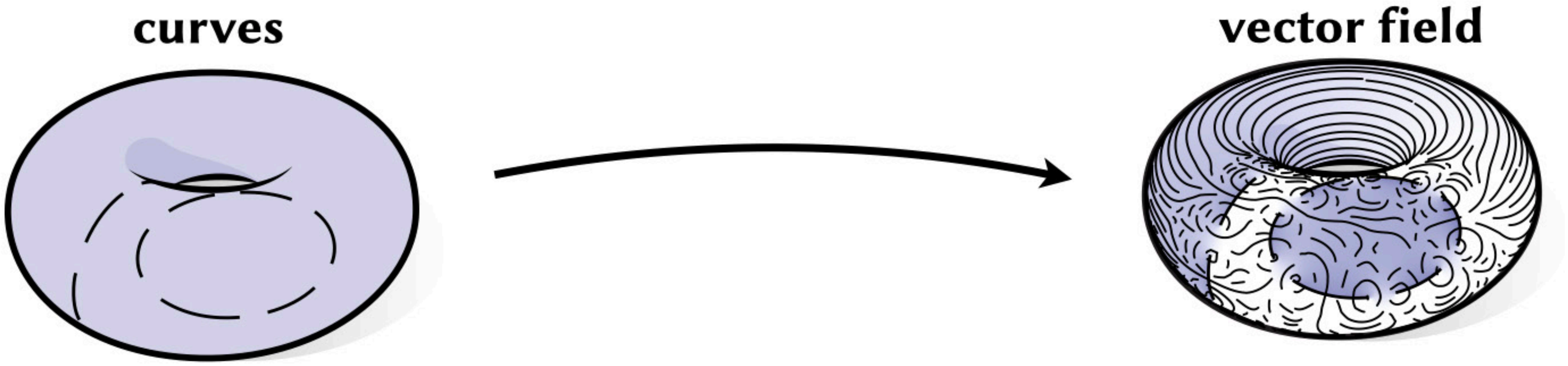

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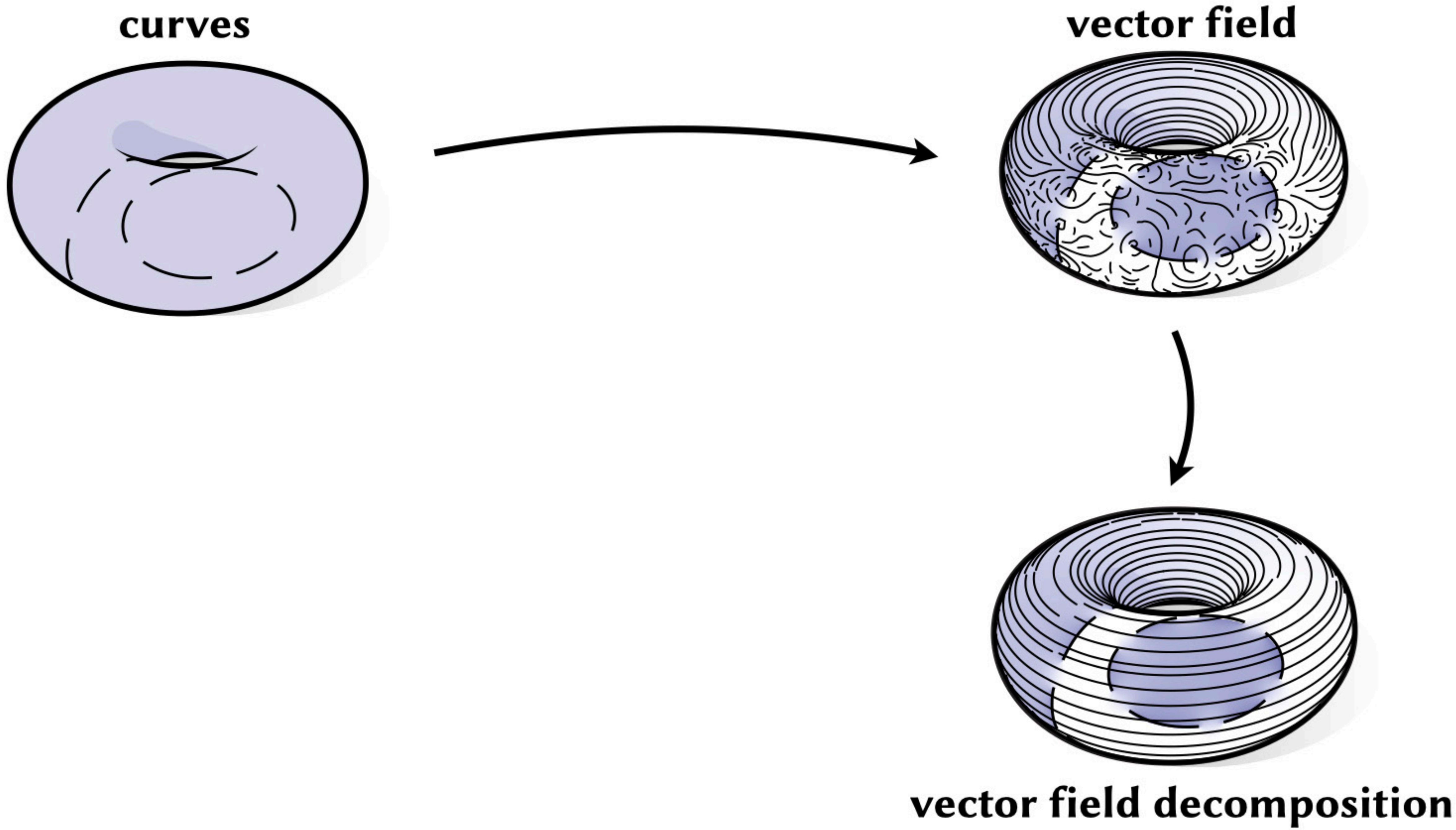

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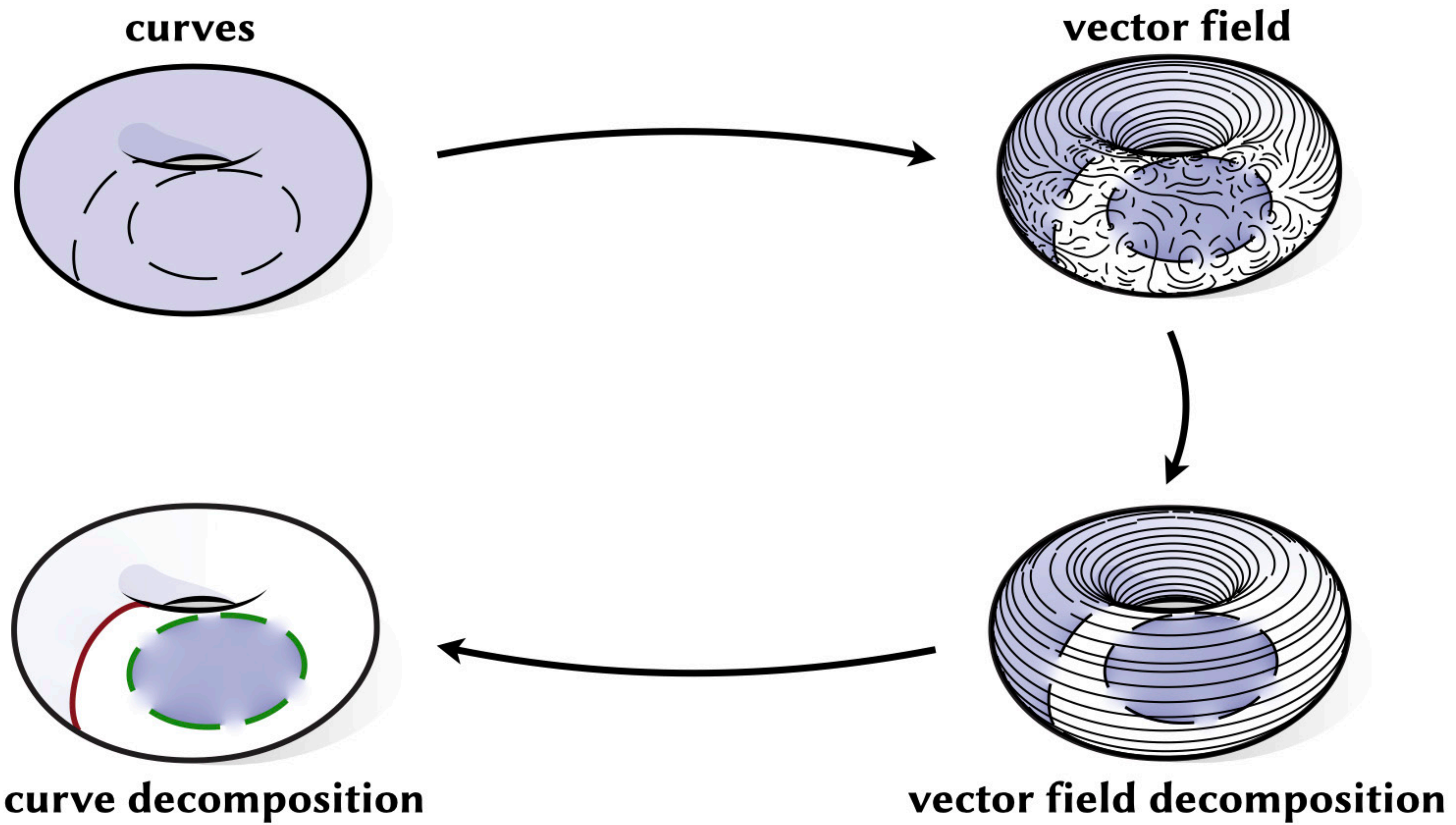

In more detail…

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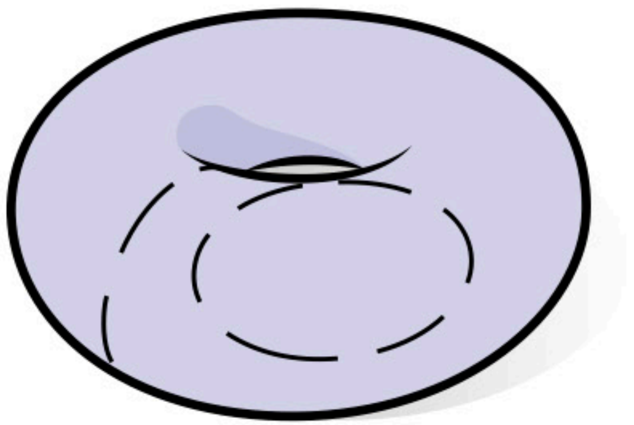
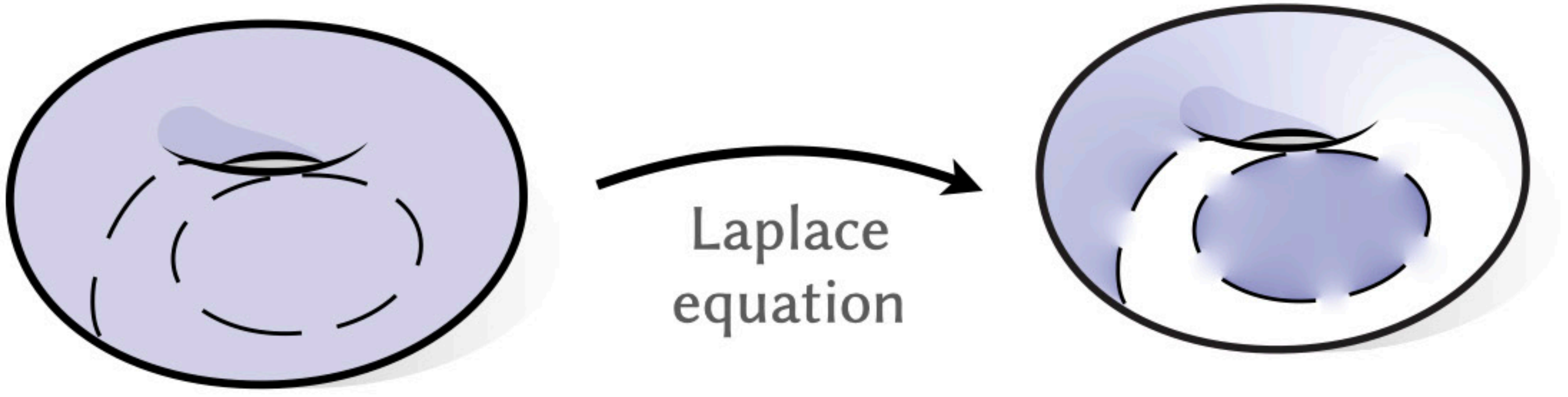
input curves



In more detail…

input curves **harmonic function with jumps**



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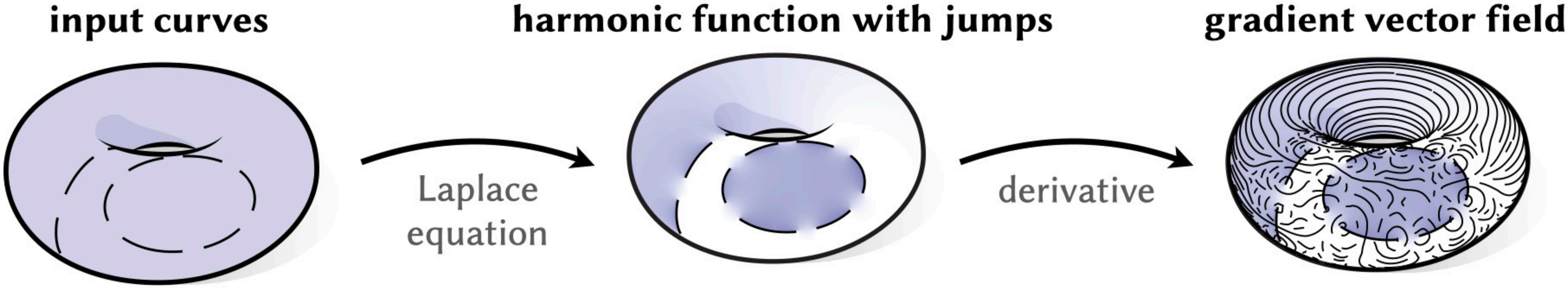

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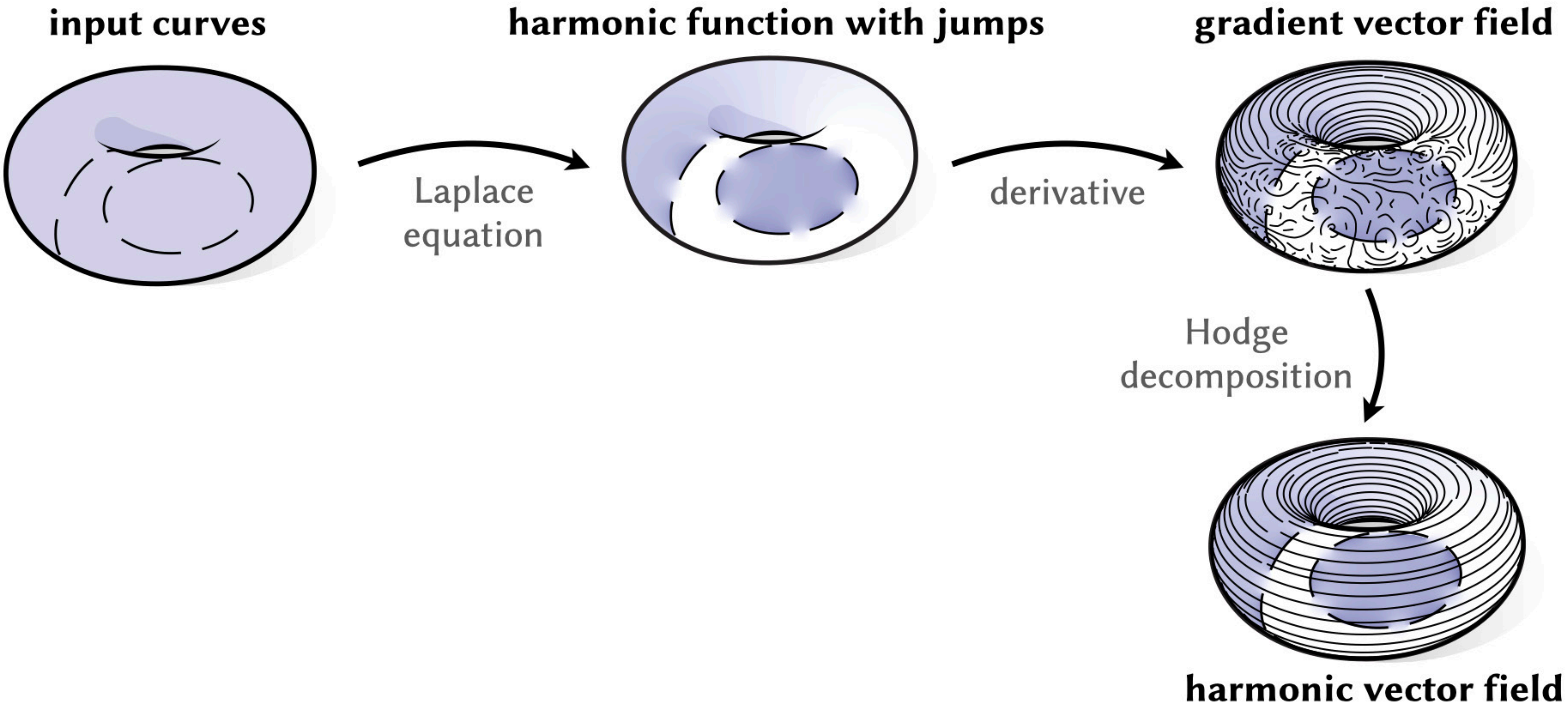

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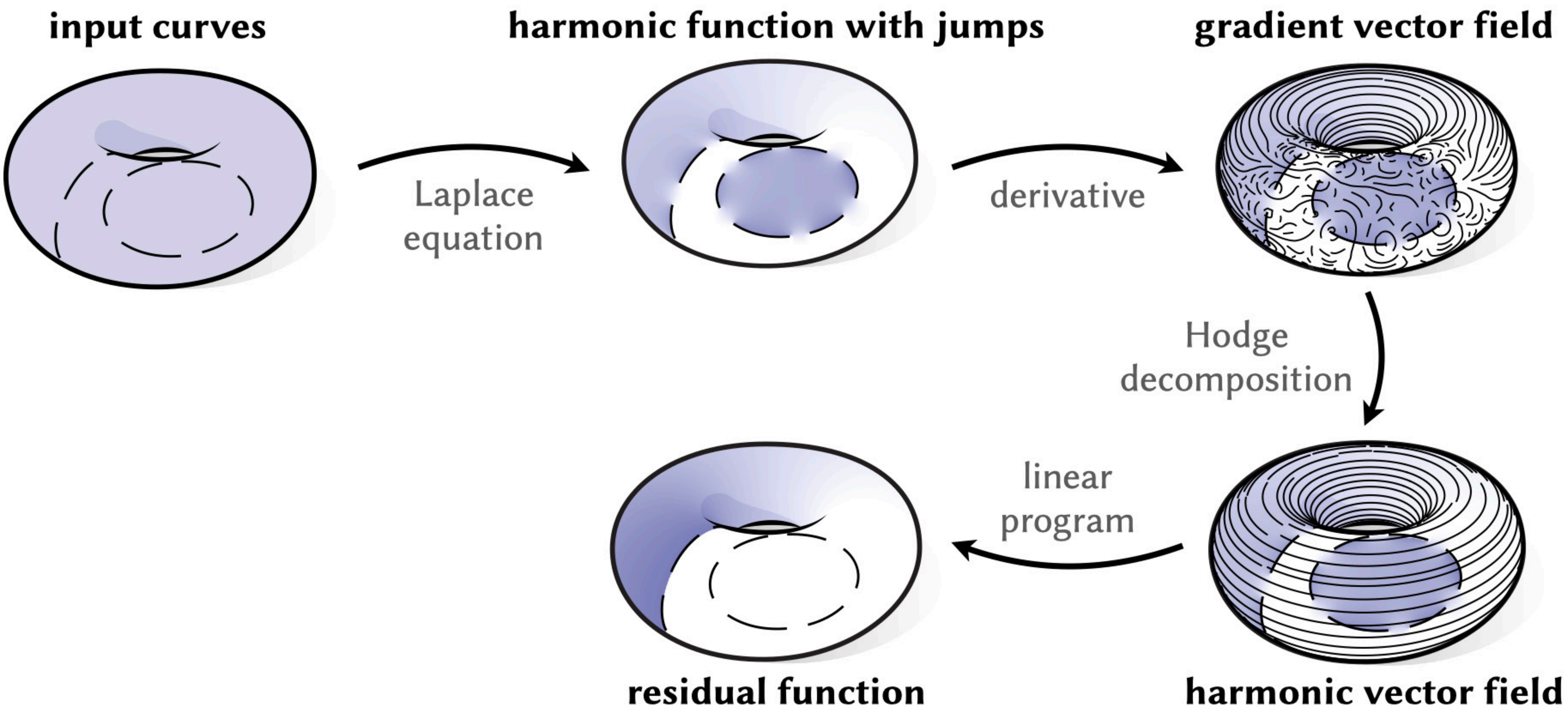

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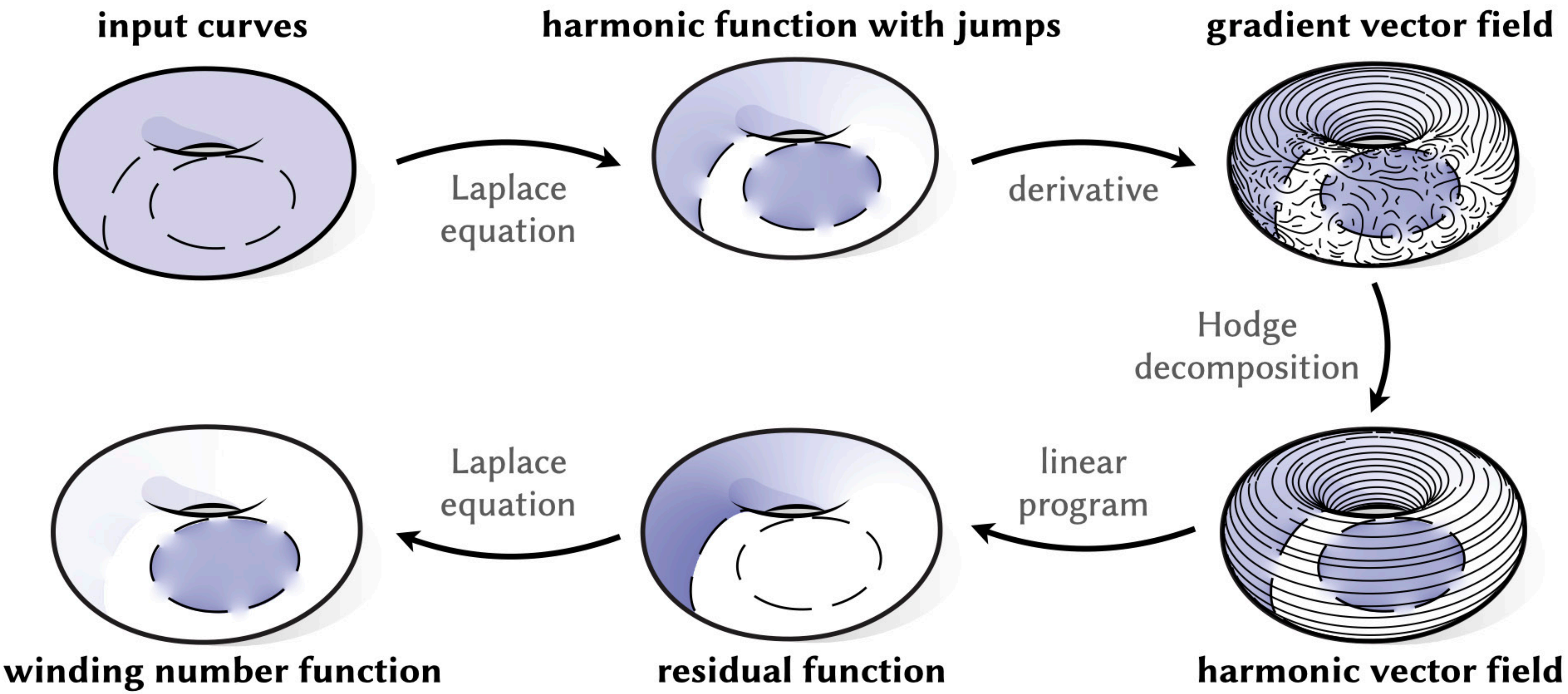

Why harmonic functions with jumps?

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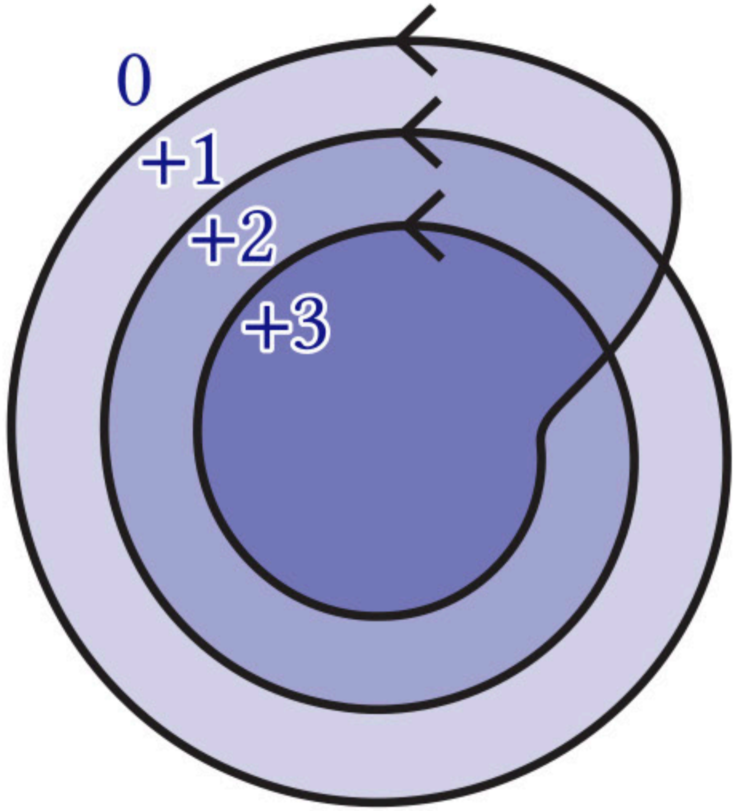

winding number

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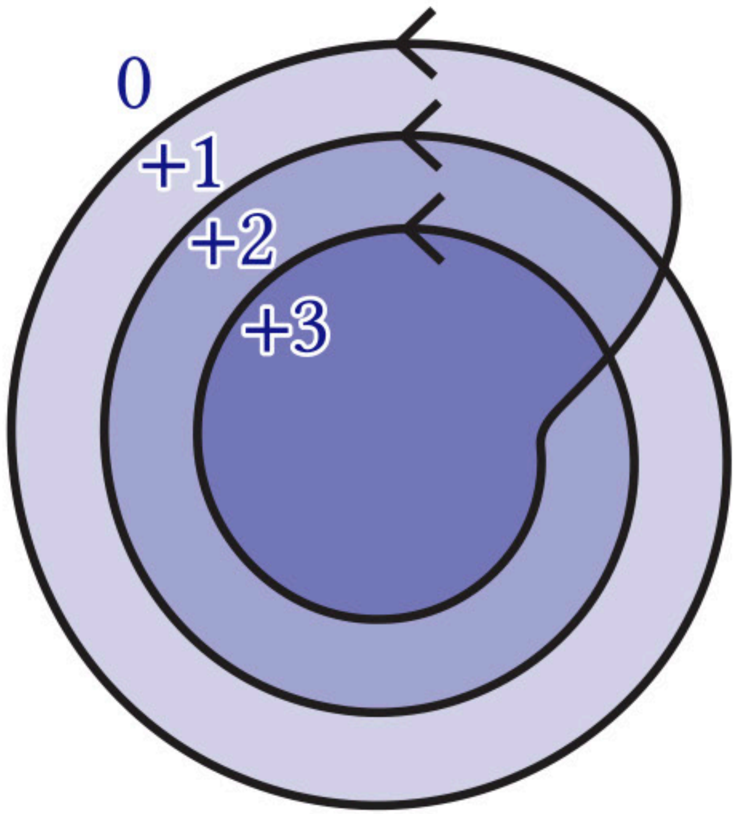

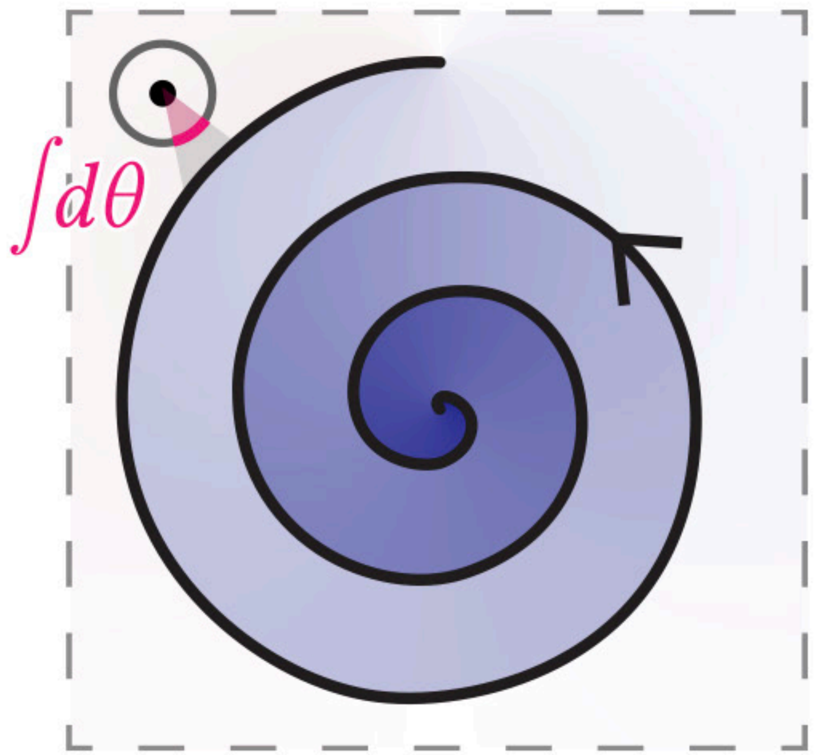

winding number $\subset$ solid angle

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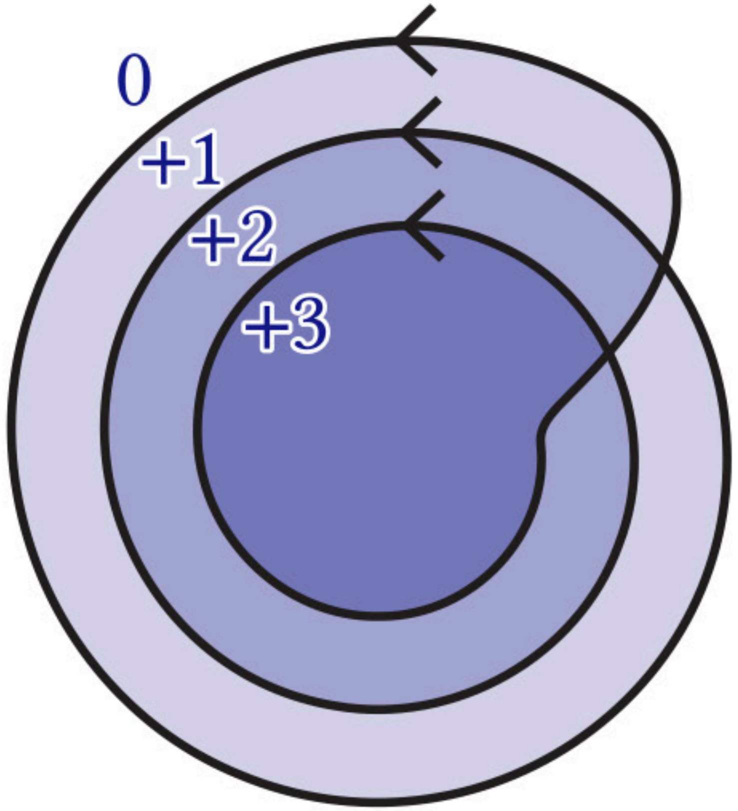
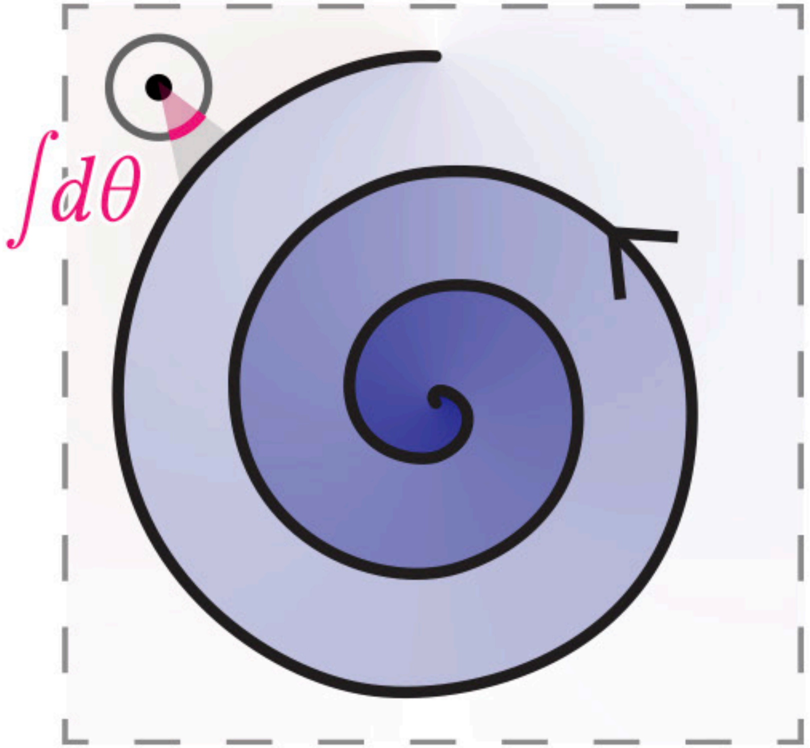
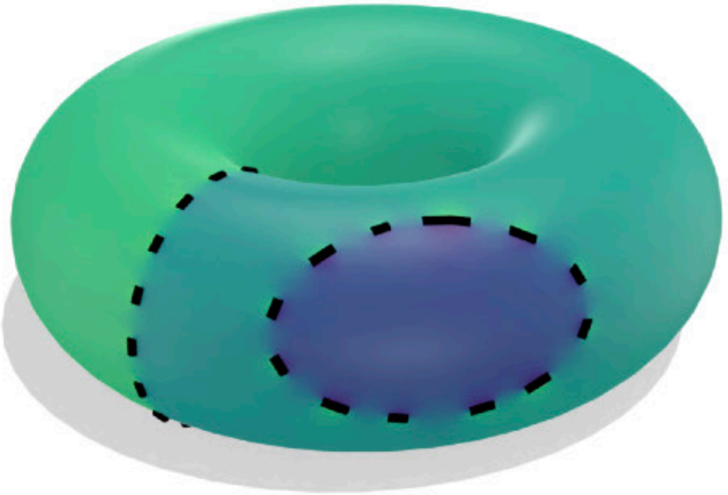






winding number ⊂ solid angle ⊂ jump harmonic functions

Jump Laplace equation

$$\Delta u = 0, \qquad \text{on } M \setminus \Gamma,$$

The jump problem for the Laplace equation. Krutitskii (2001)

Jump Laplace equation

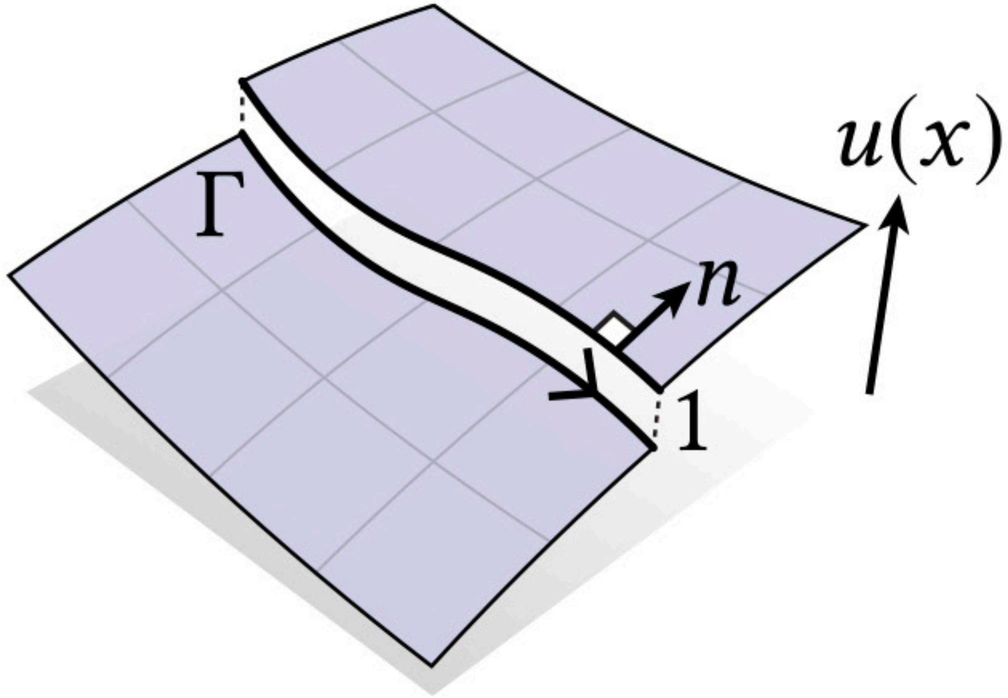
harmonic $$\Delta u = 0, \quad \text{on } M \setminus \Gamma,$$

The jump problem for the Laplace equation. Krutitskii (2001)

Jump Laplace equation

harmonic

$$\begin{aligned} \Delta u &= 0, & \text{on } M \setminus \Gamma, \\ u^+ - u^- &= 1, & \text{on } \Gamma, \end{aligned}$$



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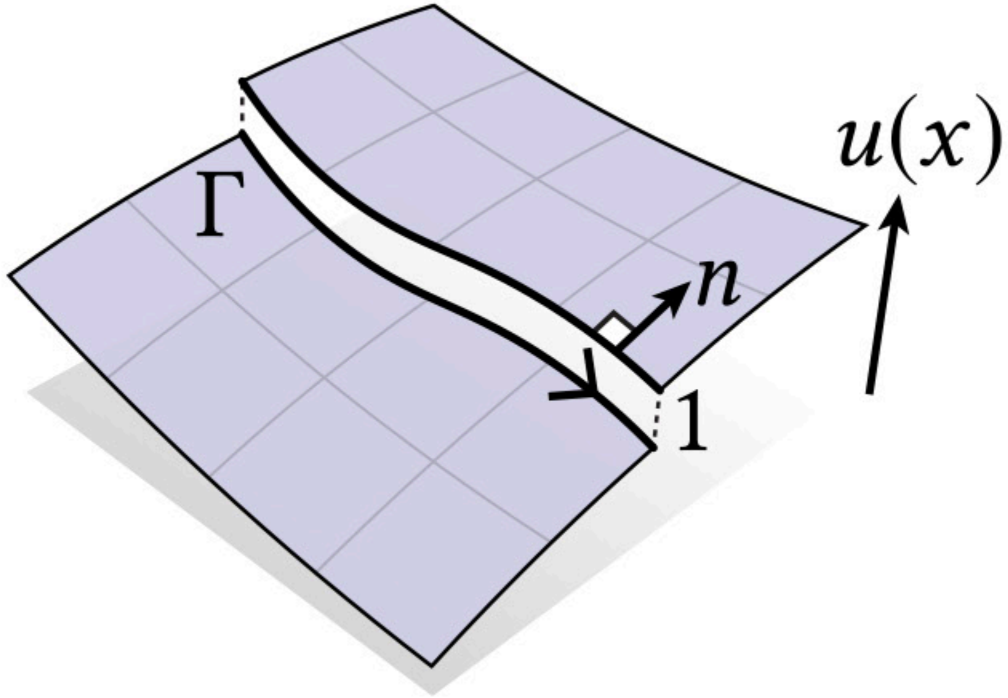
harmonic	Δu	$=$	$0,$	on $M \setminus \Gamma,$
jump	$u^+ - u^-$	$=$	$1,$	on $\Gamma,$



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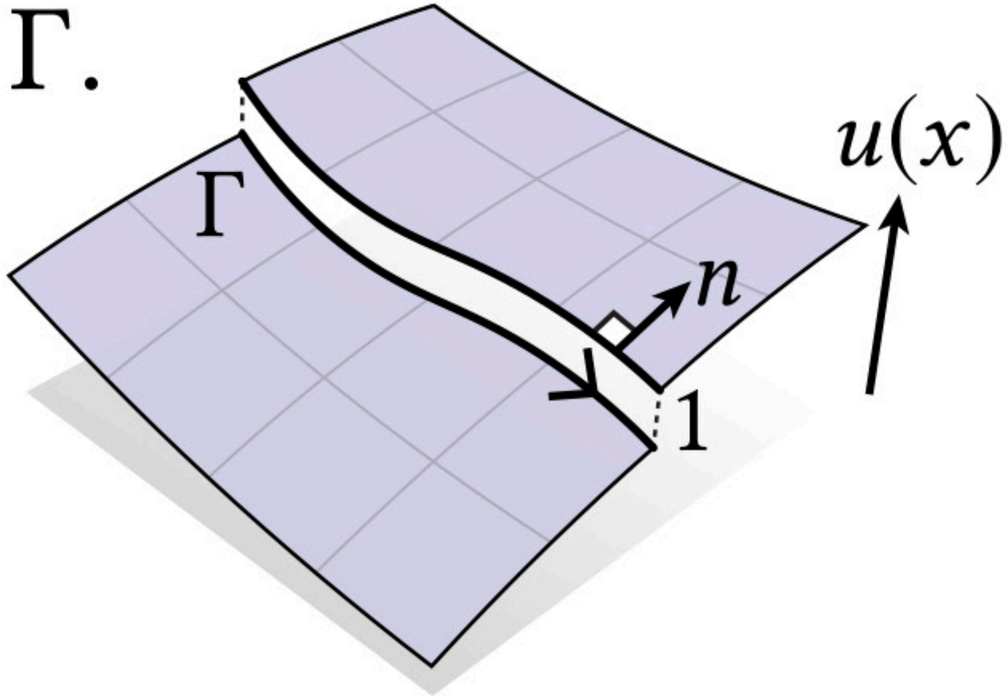
Jump Laplace equation

harmonic	Δu	$=$	$0,$	on $M \setminus \Gamma$,
jump	$u^+ - u^-$	$=$	$1,$	on Γ,
	$\partial u^+/\partial n$	$=$	$\partial u^-/\partial n,$	on Γ.



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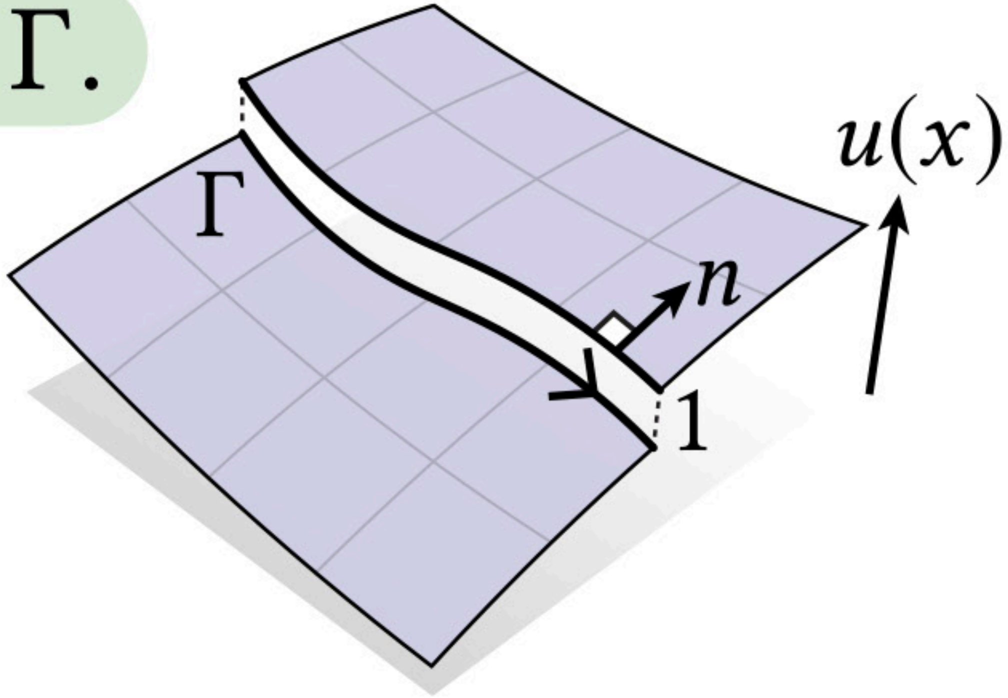
Jump Laplace equation

harmonic	Δu	$=$	$0,$	on $M \setminus \Gamma$,
jump	$u^+ - u^-$	$=$	$1,$	on Γ,
compatibility	$\partial u^+ / \partial n$	$=$	$\partial u^- / \partial n,$	on Γ.

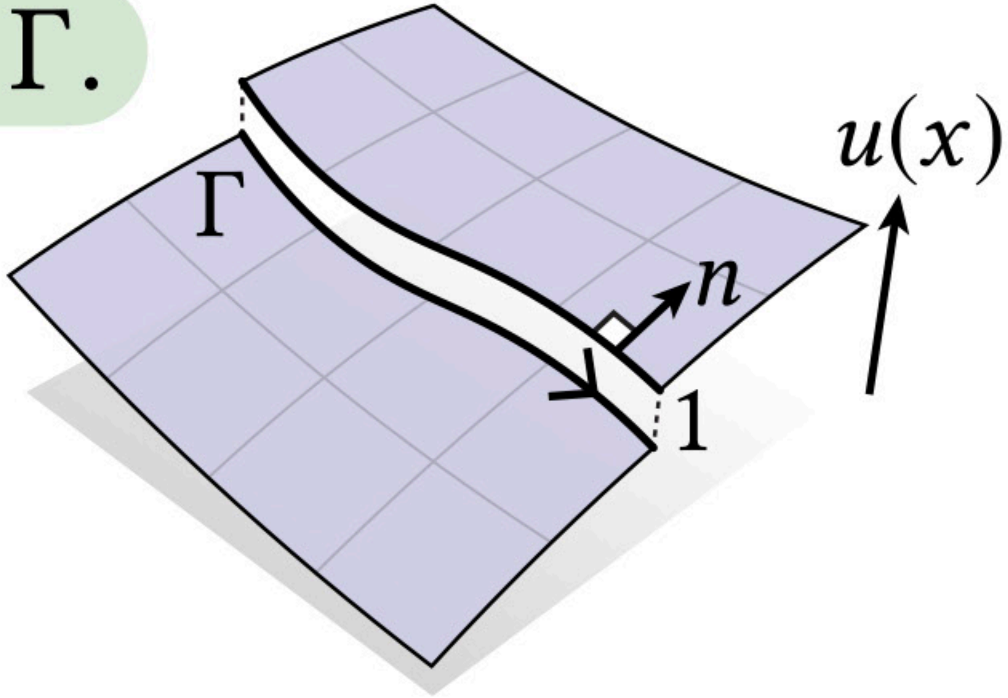


The jump problem for the Laplace equation. Krutitskii (2001)

Jump Laplace equation

harmonic	Δu	$=$	$0,$	on $M \setminus \Gamma$,
jump	$u^+ - u^-$	$=$	$1,$	on Γ,
compatibility	$\partial u^+ / \partial n$	$=$	$\partial u^- / \partial n,$	on Γ.

sparse $|V| \times |V|$ linear system



The jump problem for the Laplace equation. Krutitskii (2001)

If domain has trivial topology:

jump harmonic function

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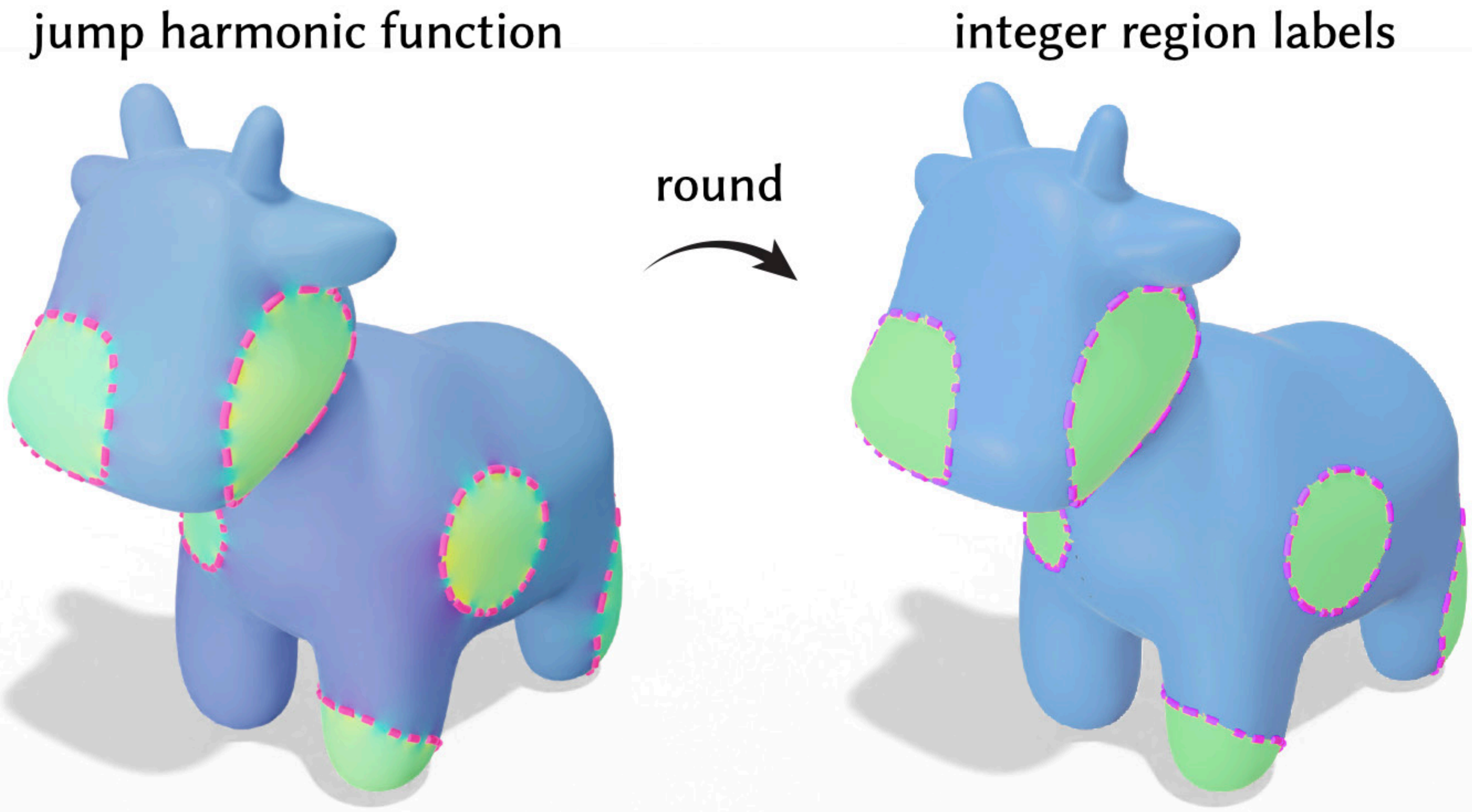
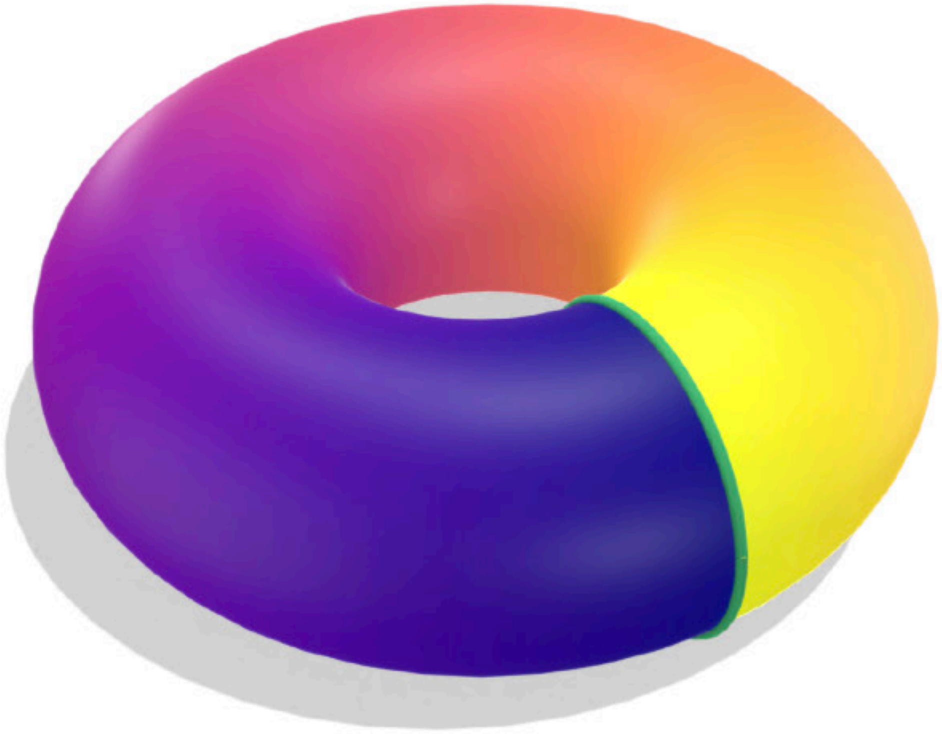
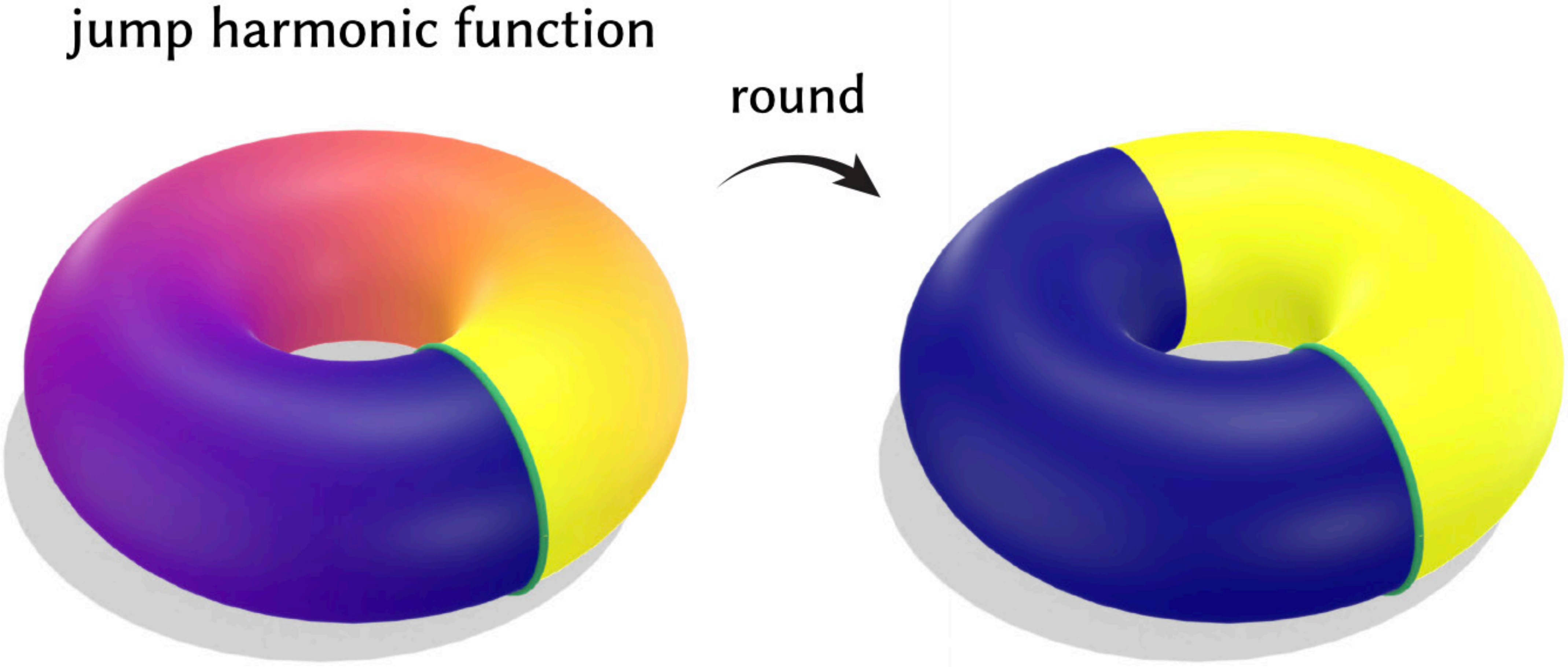


If domain has nontrivial topology:

jump harmonic function



If domain has <u>nontrivial</u> topology:



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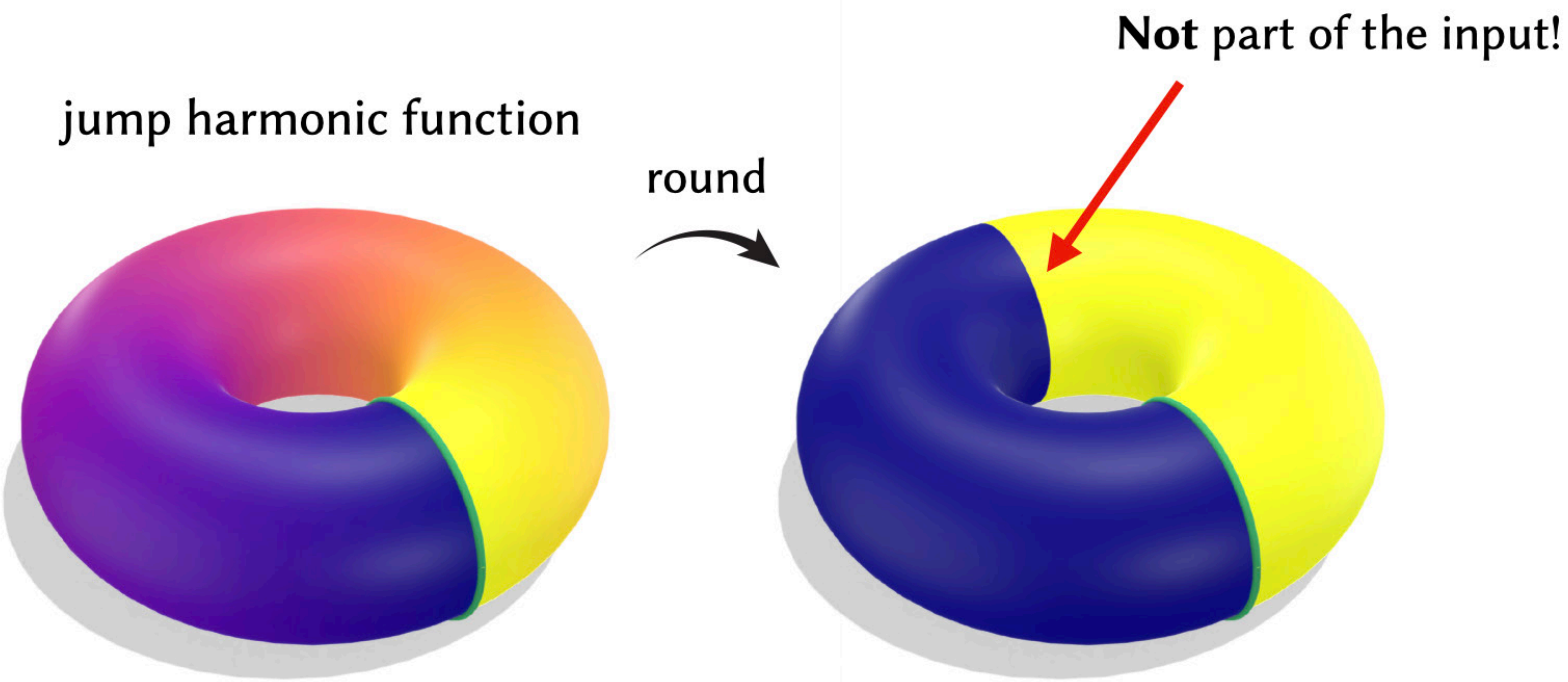
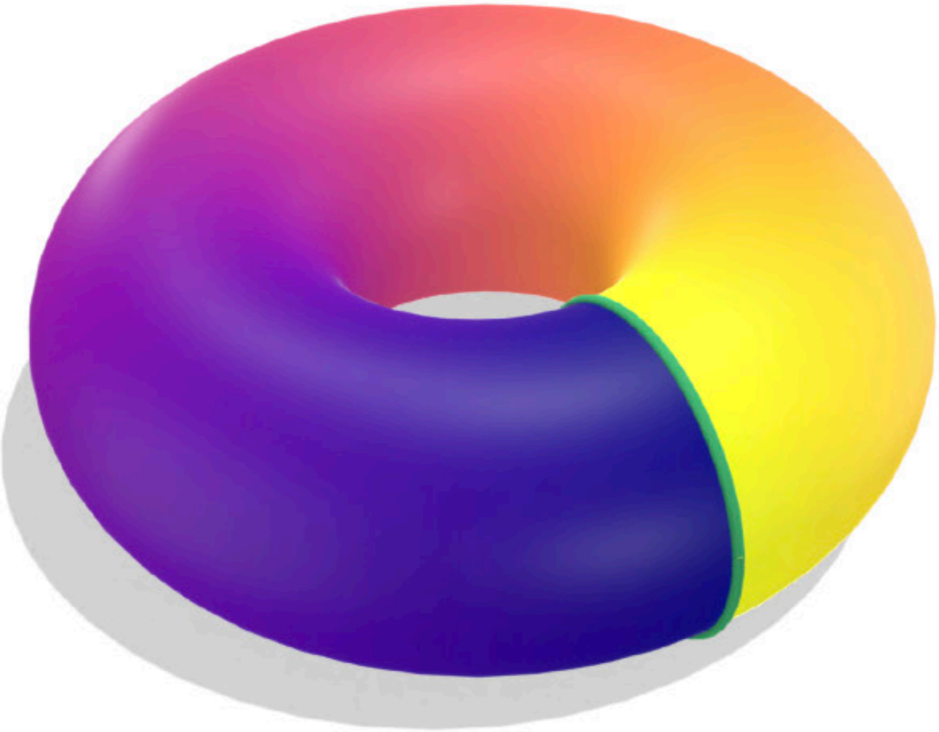


Differentiating jump harmonic functions

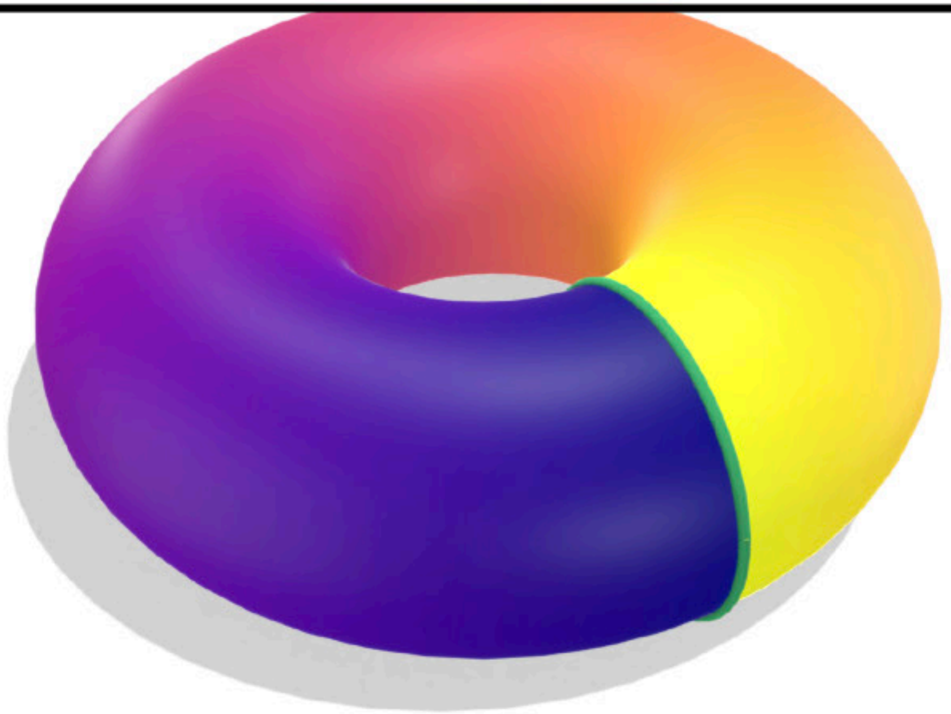


u

Differentiating jump harmonic functions

$$
\begin{aligned}
\Delta u &= 0, \quad \text{on } M \setminus \Gamma, \\
u^+ - u^- &= 1, \\
\partial u^+/\partial n &= \partial u^-/\partial n,
\end{aligned}
$$

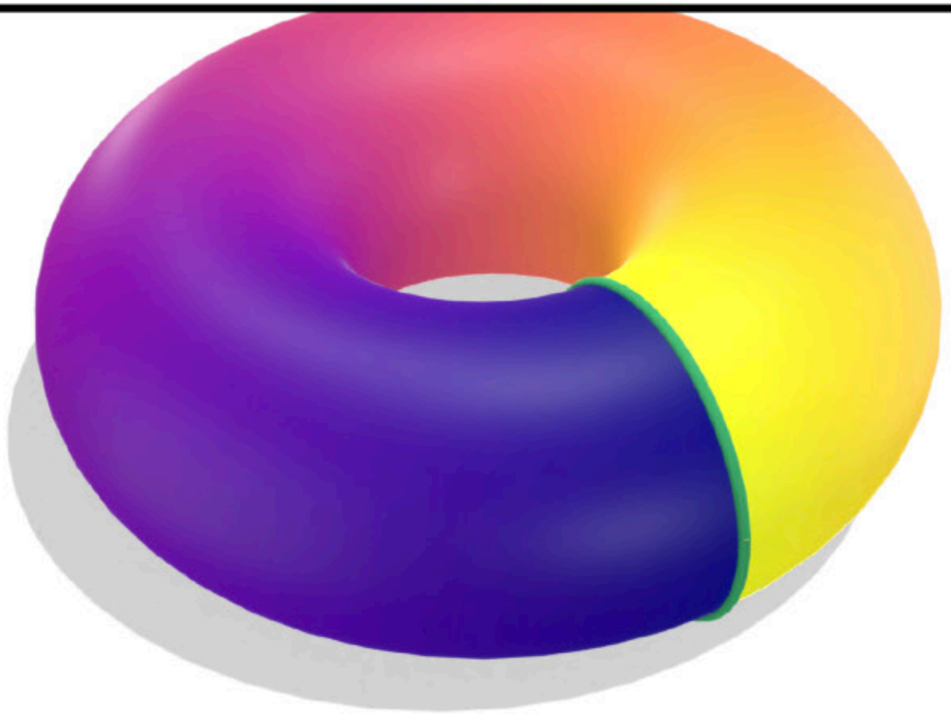
compatibility



u

Differentiating jump harmonic functions

$$\begin{aligned} \Delta u &= 0, \quad \text{on } M \setminus \Gamma, \\ u^+ - u^- &= 1, \\ \text{compatibility} \quad \partial u^+/\partial n &= \partial u^-/\partial n, \end{aligned}$$



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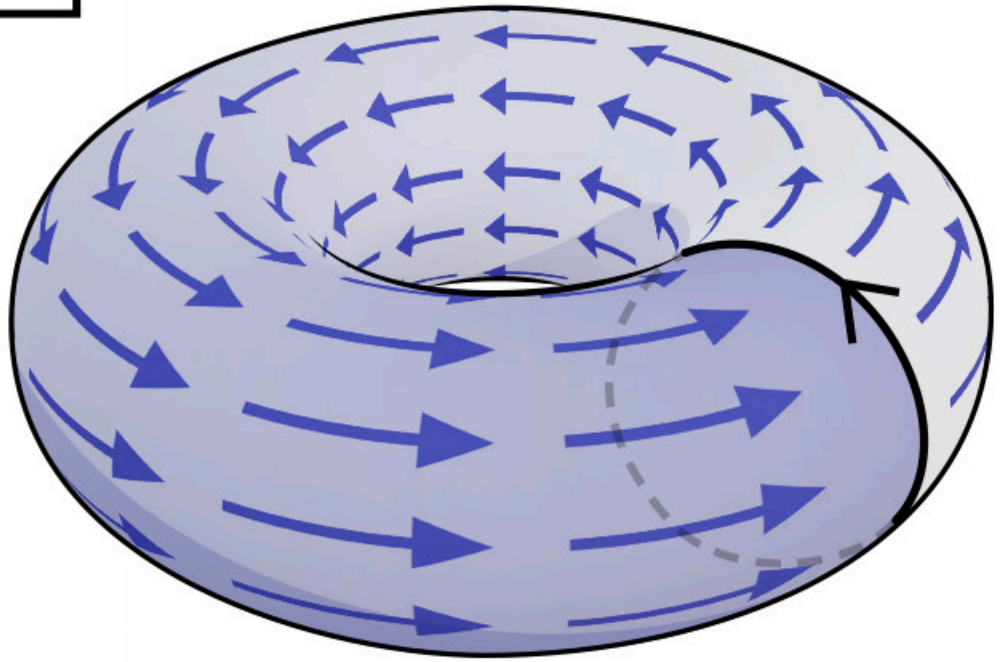
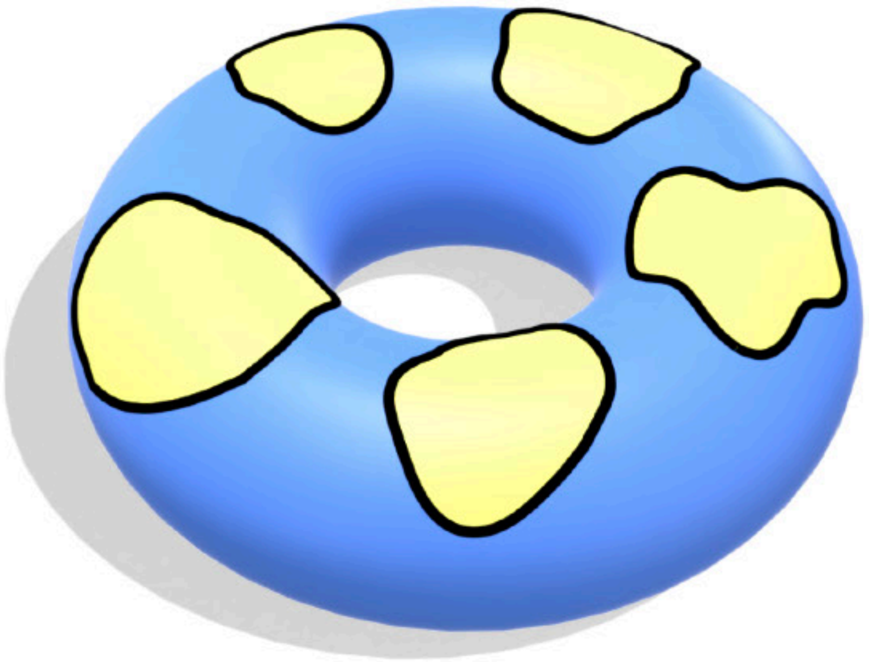


$\omega :=$ derivative of u

What does the derivative tell us?

What does the derivative tell us?

If all curves are **bounding**,
the derivative is **zero**.



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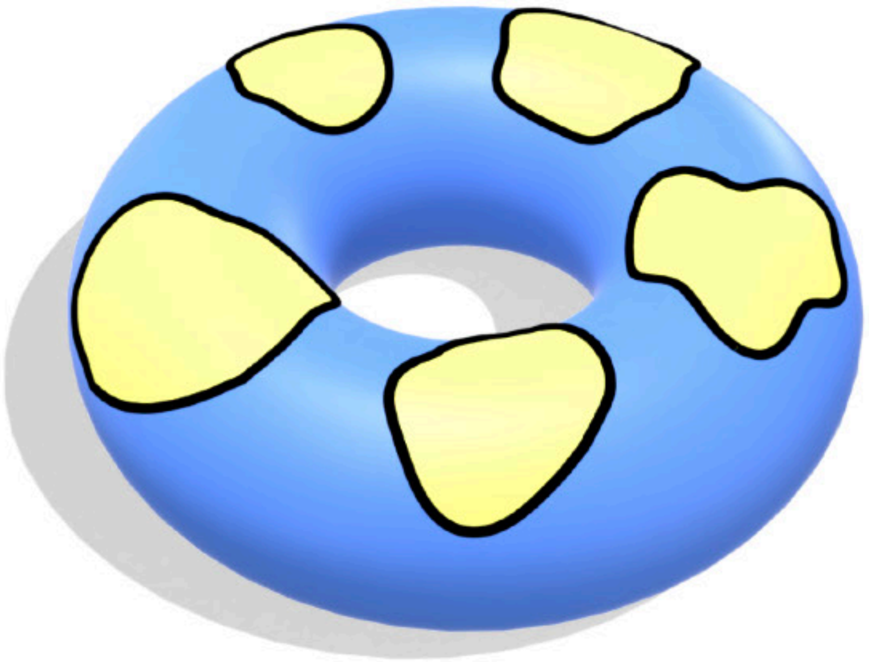
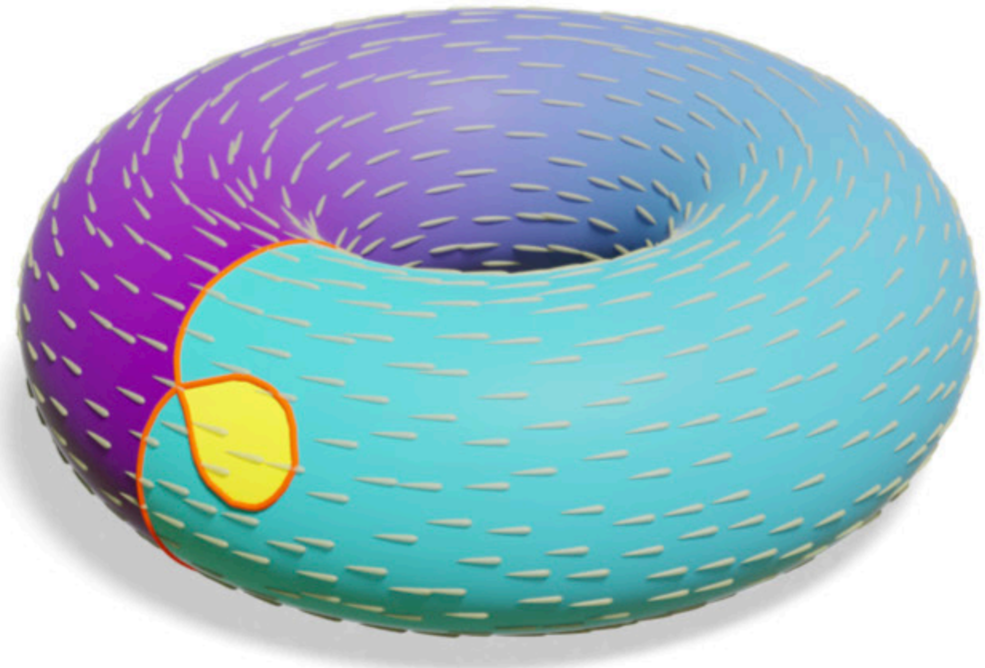


If some curves are **nonbounding**,
the derivative is nonzero and **harmonic**.



If the curve is broken, find closest harmonic field

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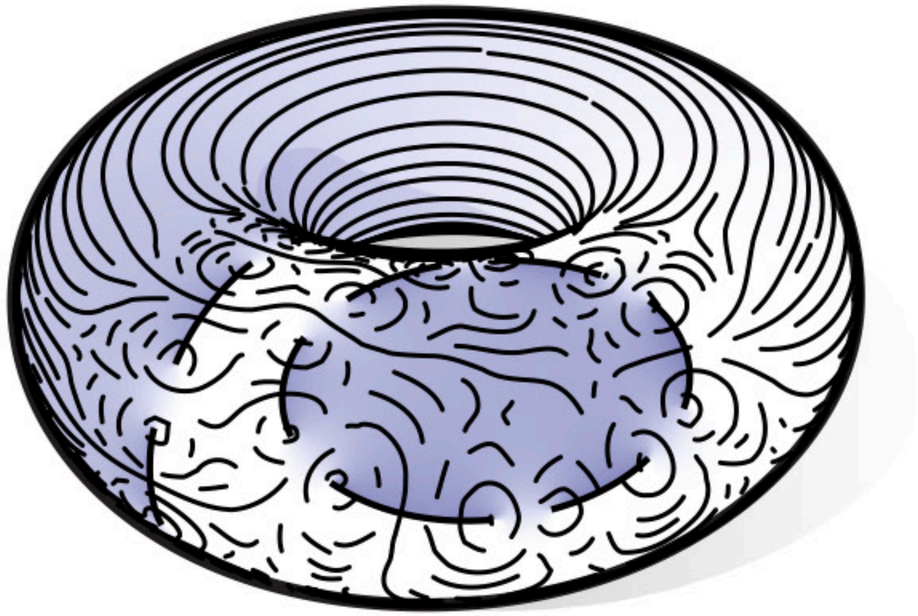
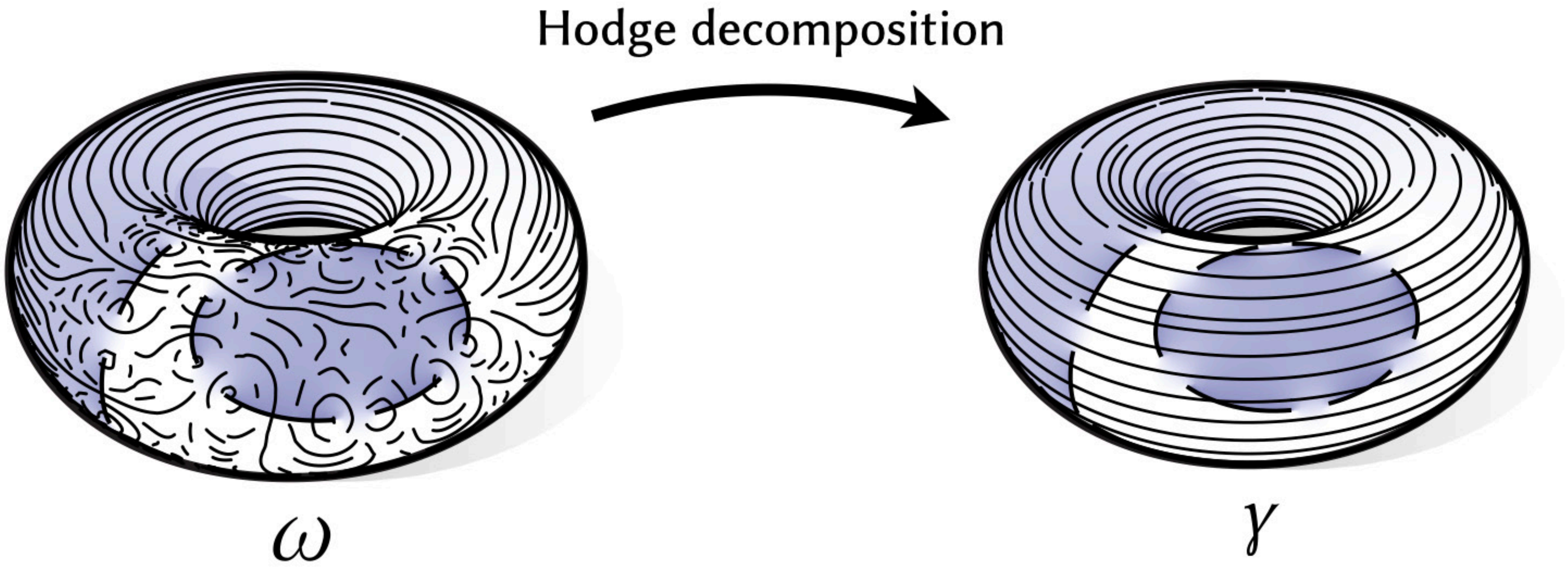


ω

If the curve is broken, find closest harmonic field



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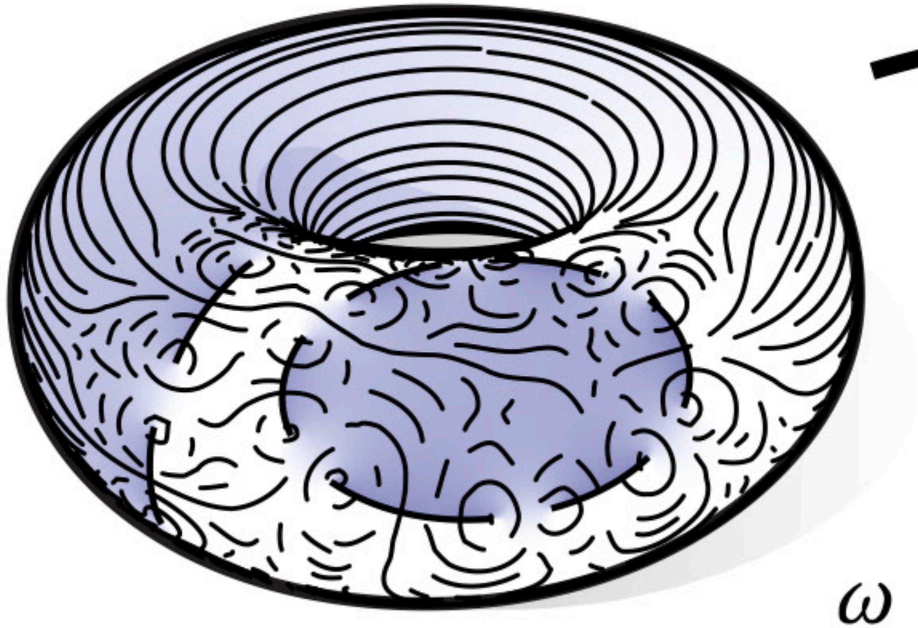


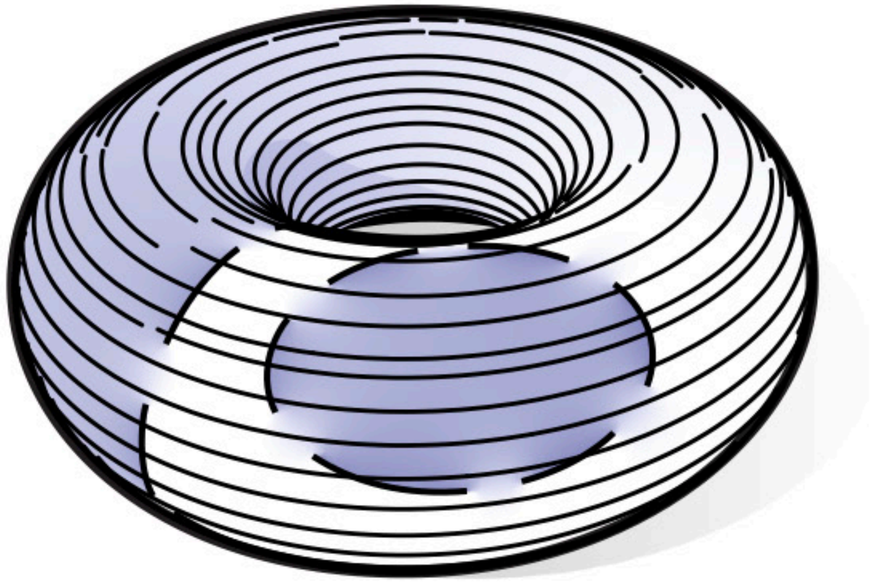
ω

Hodge decomposition

$$\omega = \cancel{d\alpha} + \delta\beta + \gamma$$

(Lemma, Appendix A)

γ is a **harmonic 1-form**



γ

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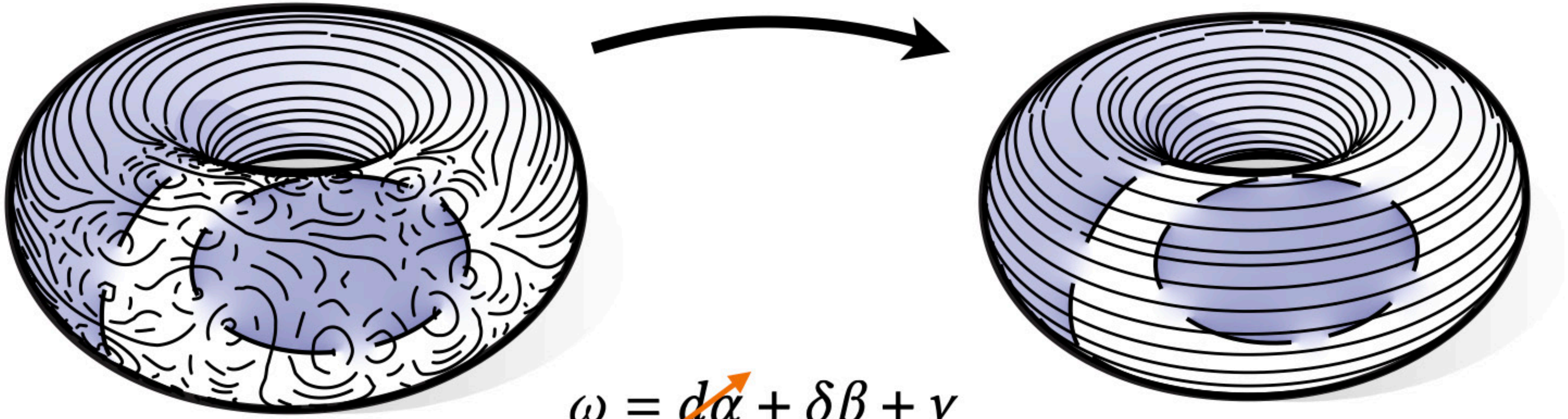
Hodge decomposition



ω $\qquad$ γ

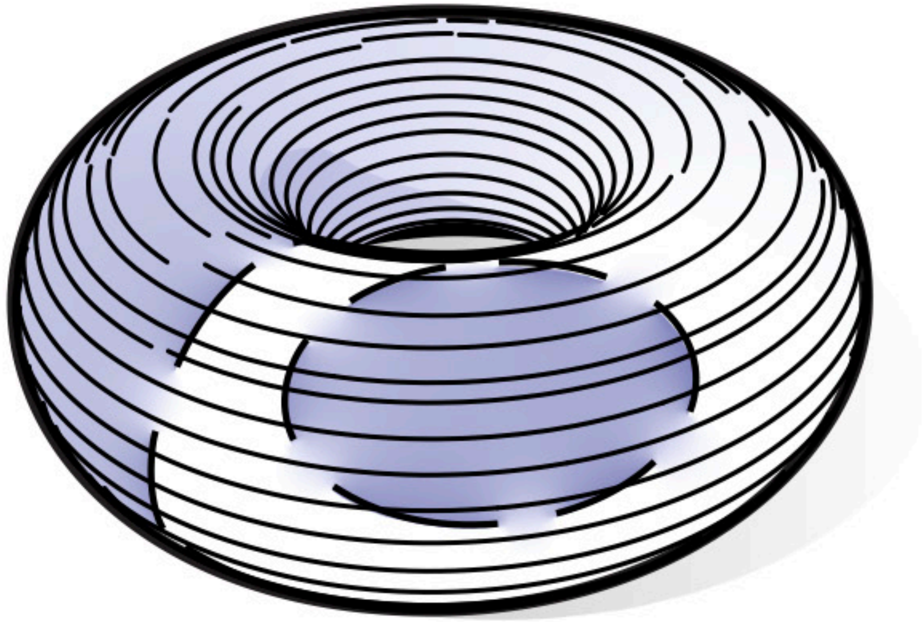
$$\omega = d\alpha + \delta\beta + \gamma$$

(Lemma, Appendix A)

γ is a **harmonic 1-form**

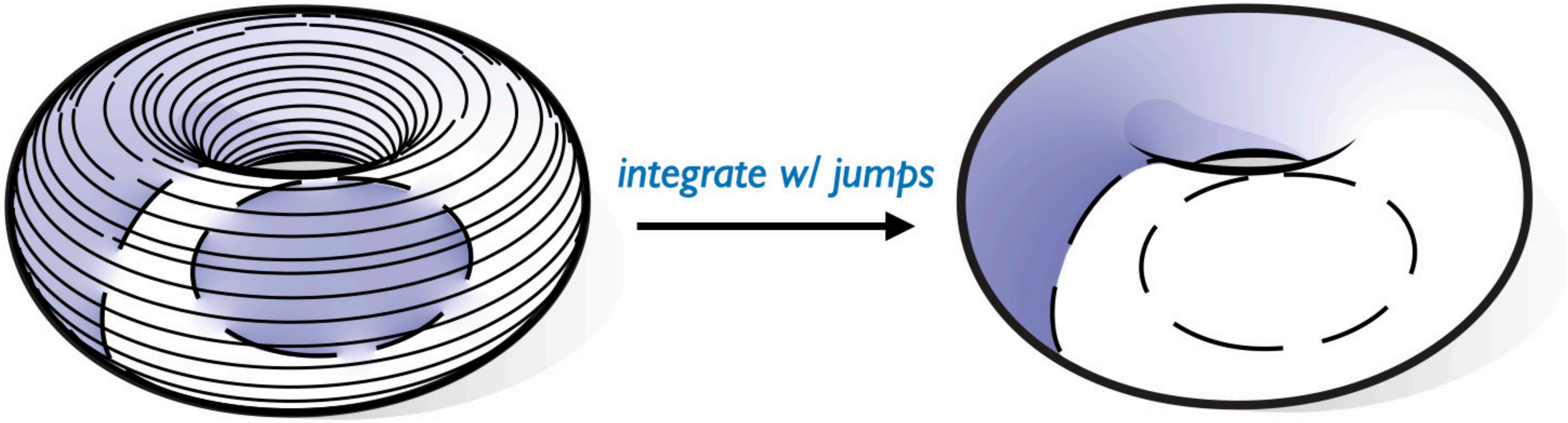
sparse $|F| \times |F|$ linear Poisson system

Use harmonic vector field to decompose curve



harmonic field

Use harmonic vector field to decompose curve



harmonic field

residual function

Linear program for the residual

penalize jumps — *smaller penalty across Γ*

$$\min_{v\colon M\to\mathbb{R}} \boxed{\int |\text{the jumps not across }\Gamma|} + \boxed{\varepsilon \int |\text{the jumps across }\Gamma|}$$

subject to $\mathcal{D}v = \gamma$ *(co)homology constraint*

$$0 \le \frac{v^+ - v^-}{u^+ - u^-} \le 1 \quad \text{on } \Gamma$$ *no extra loops*

Linear program for the residual

penalize jumps

smaller penalty across Γ

m... s $\Gamma|$

v: M...

Find shortest completion that agrees with the harmonic field

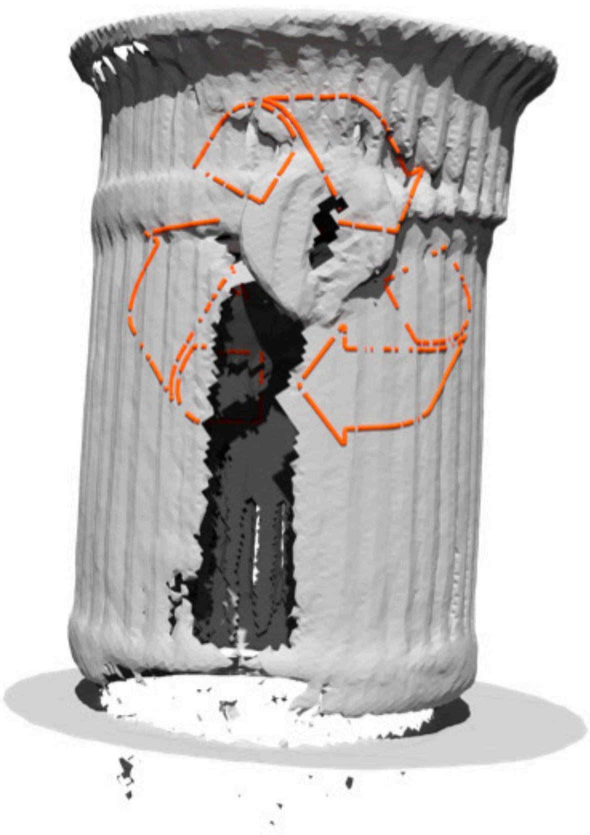
$u^+ - u^-$

sparse linear program

Results

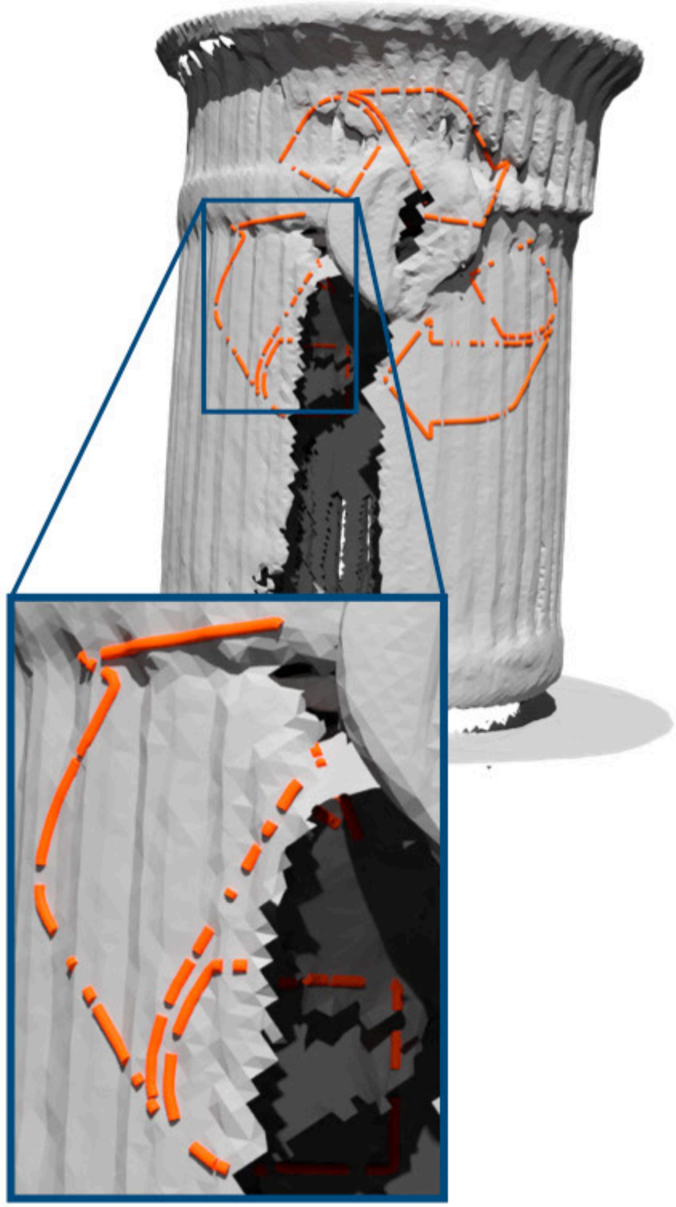
Robustness to defects in both Γ and M

Input



Robustness to defects in both Γ and M

Input



Robustness to defects in both Γ and M

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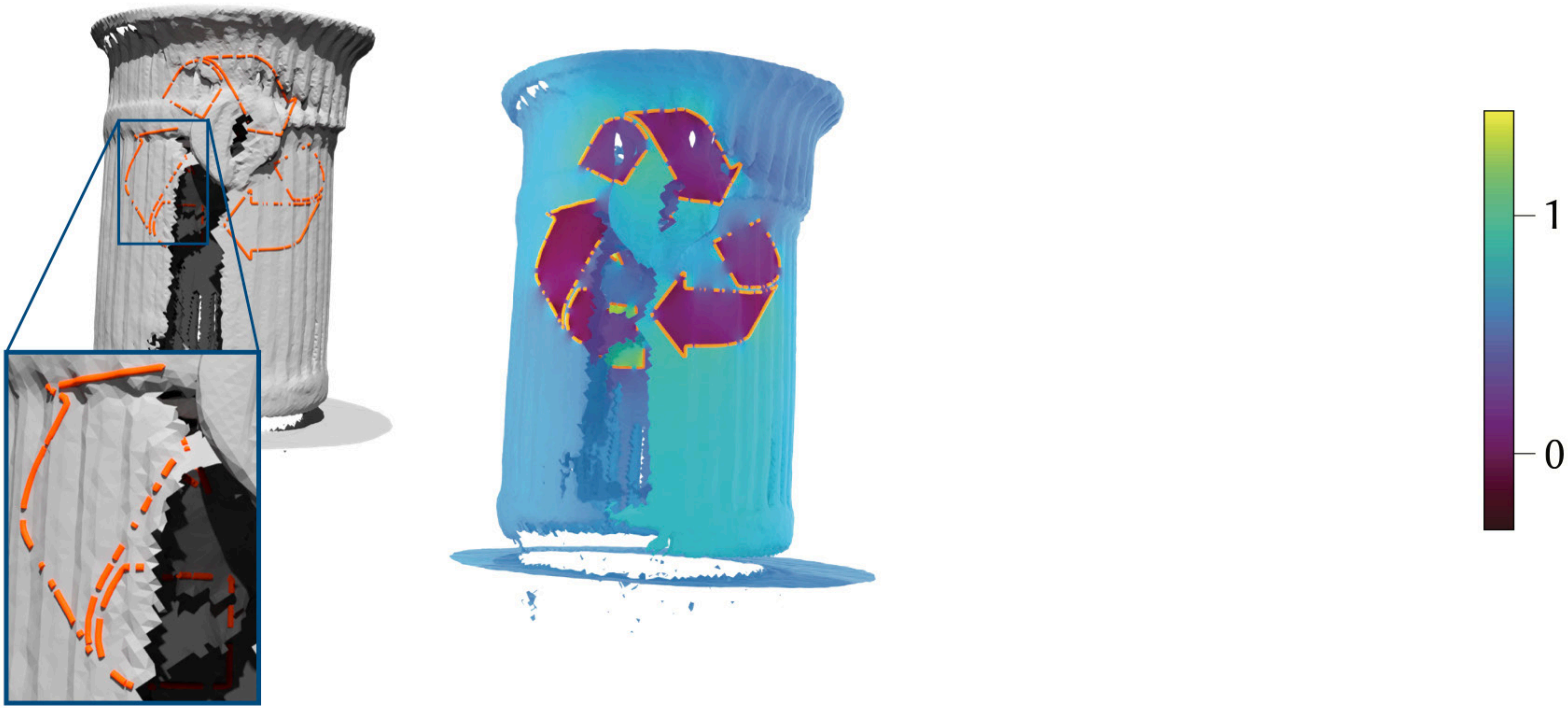

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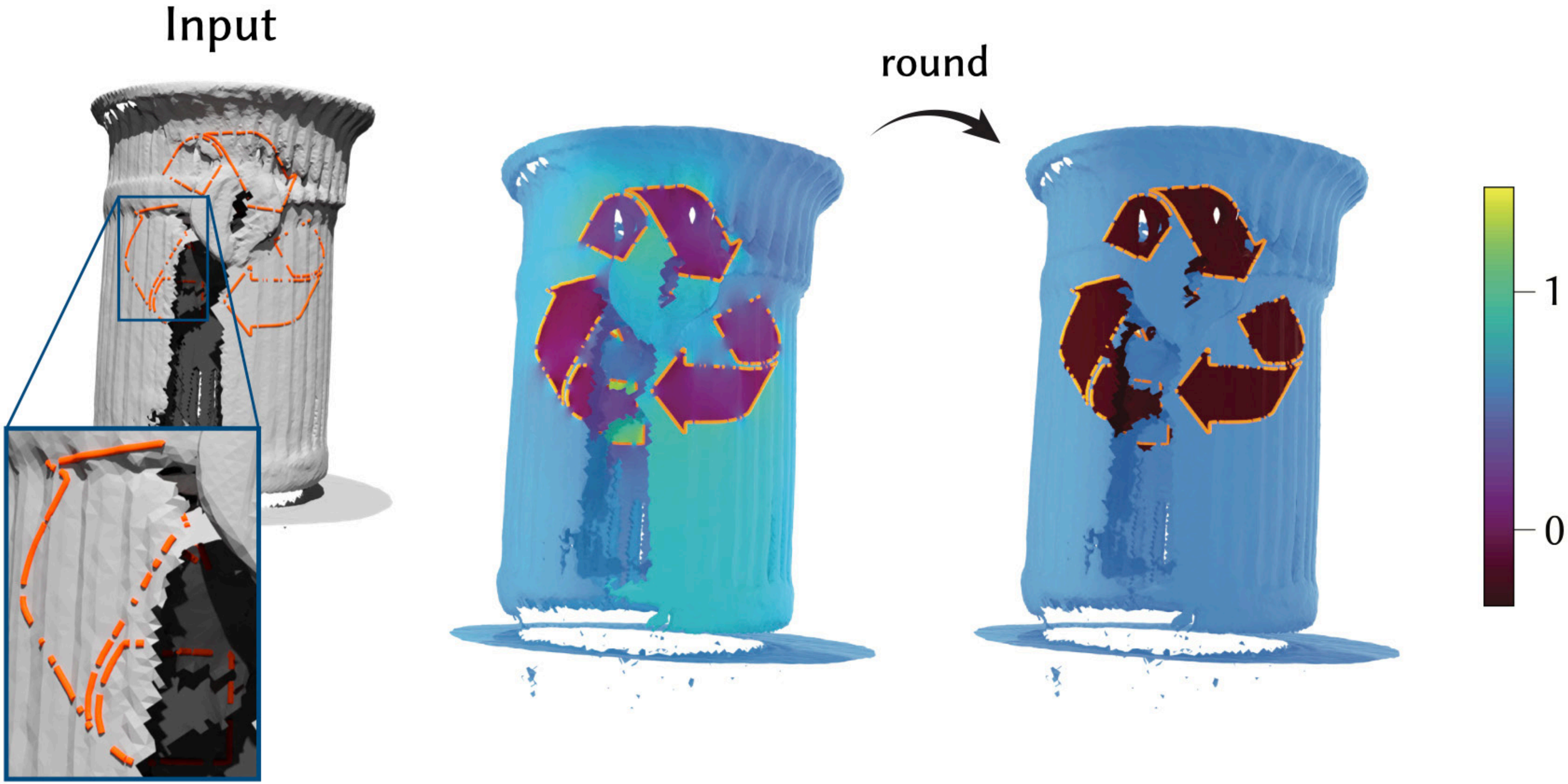
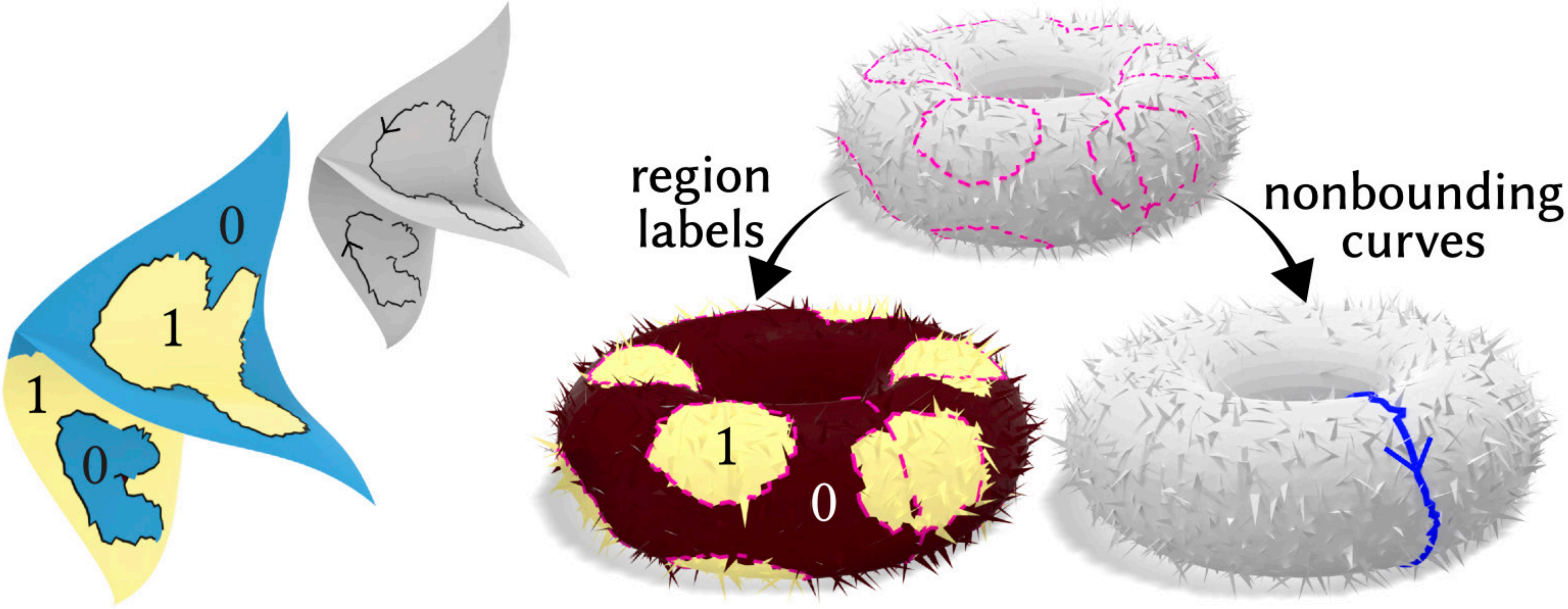
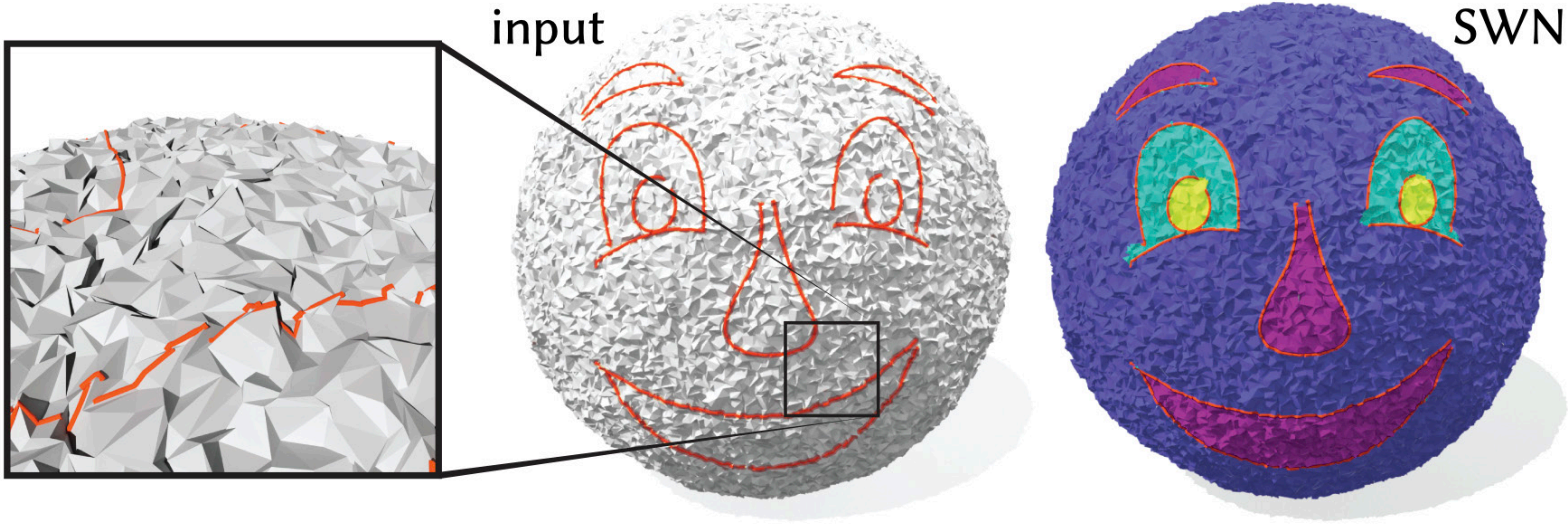


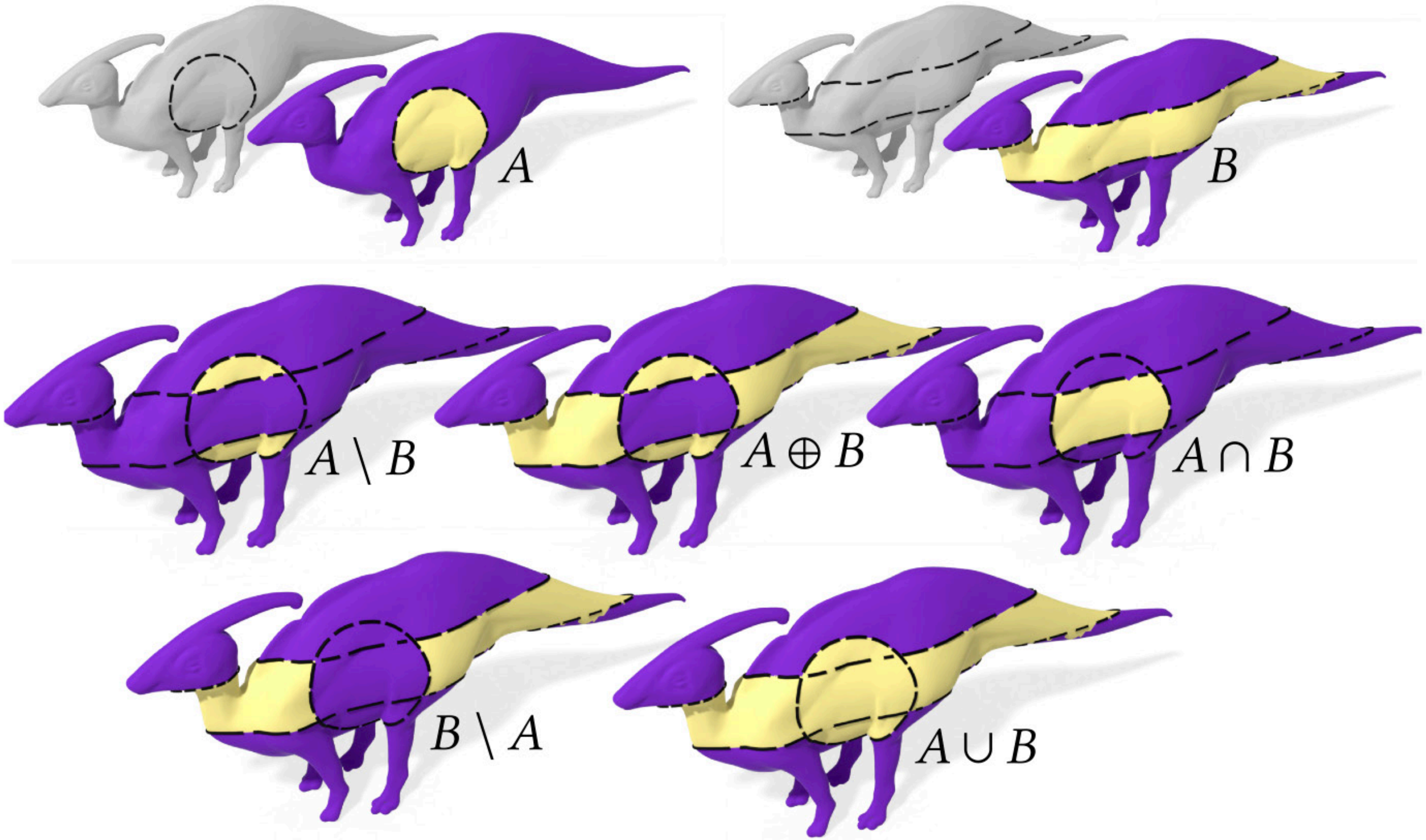
Nonmanifold meshes



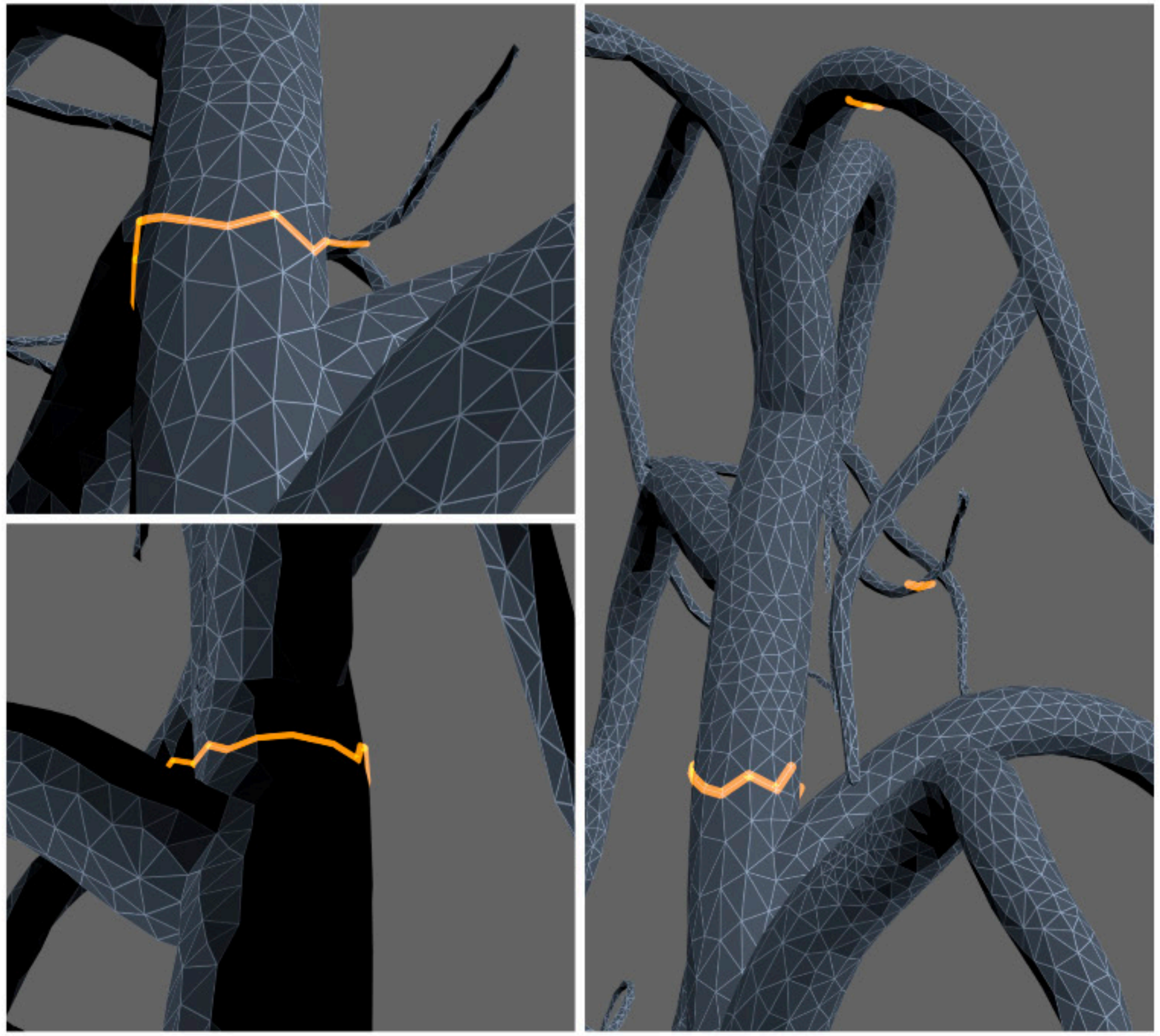
Surface painting

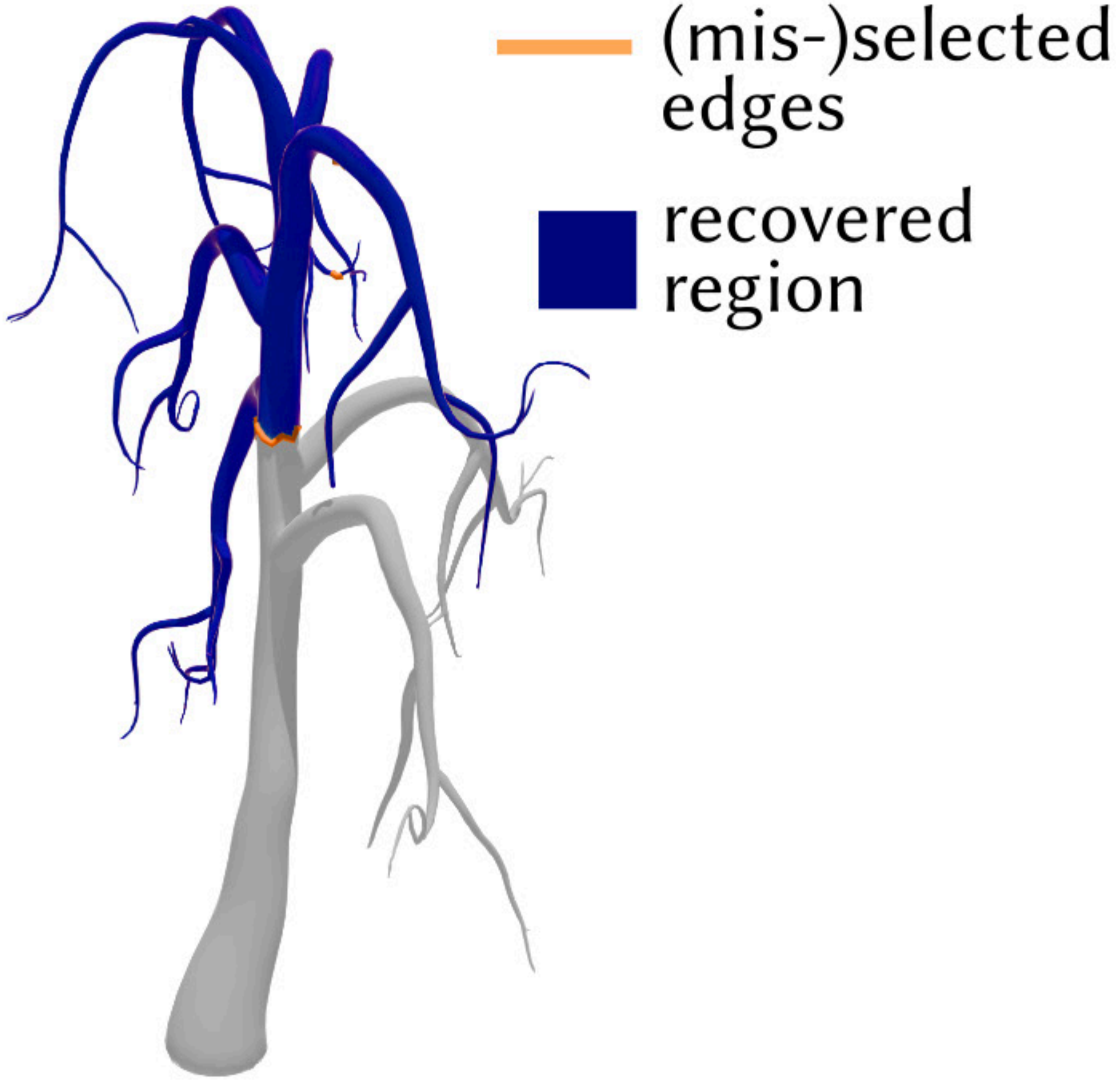


Booleans

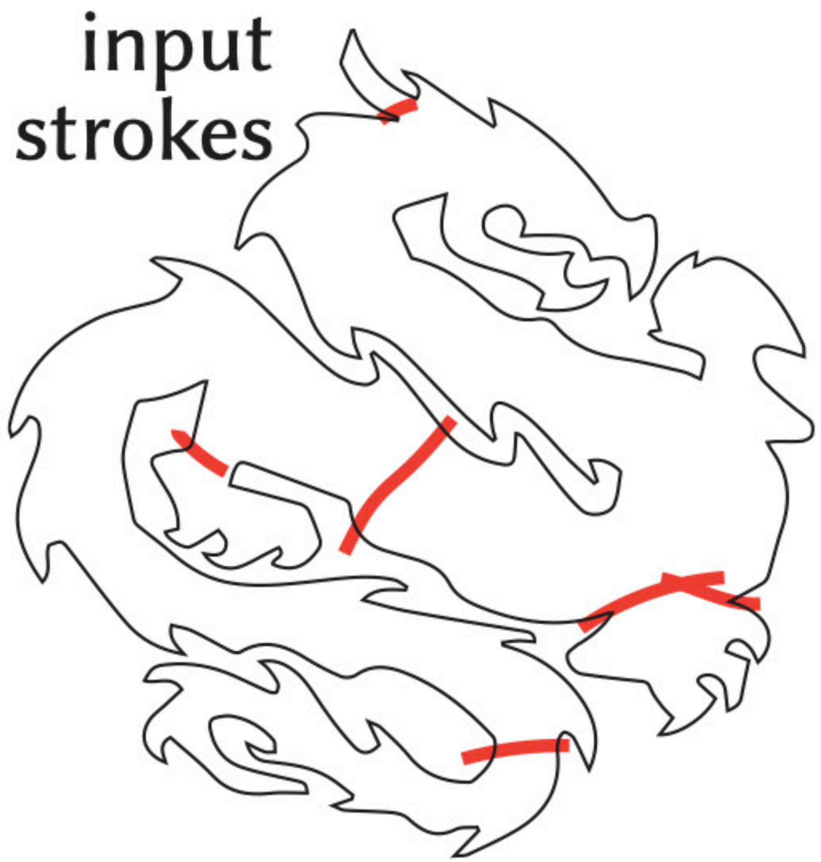


Region selection





Surface winding numbers are useful even in 2D



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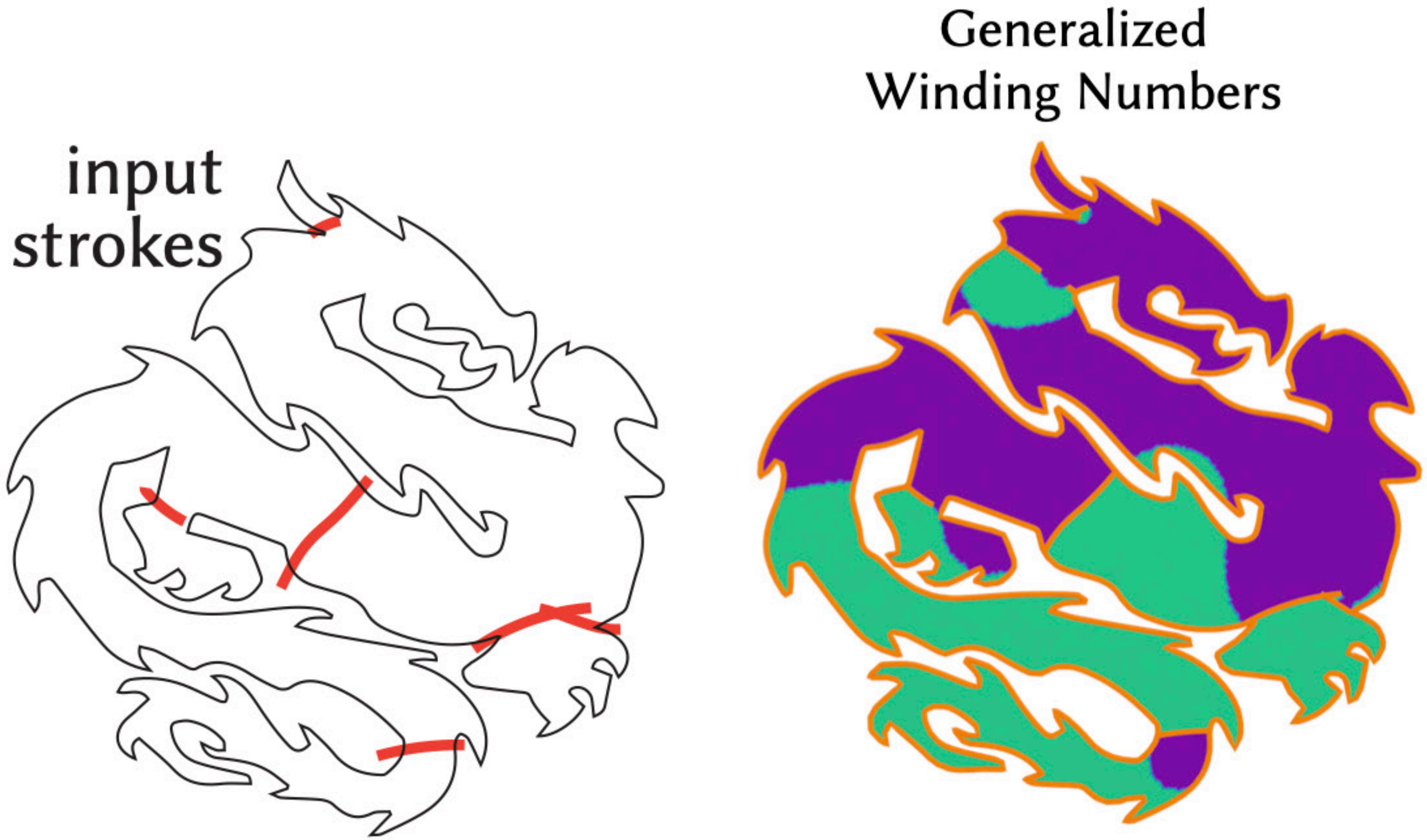

Robust Inside-Outside Segmentation using Generalized Winding Numbers. Jacobson, Kavan, Sorkine-Hornung (2013)

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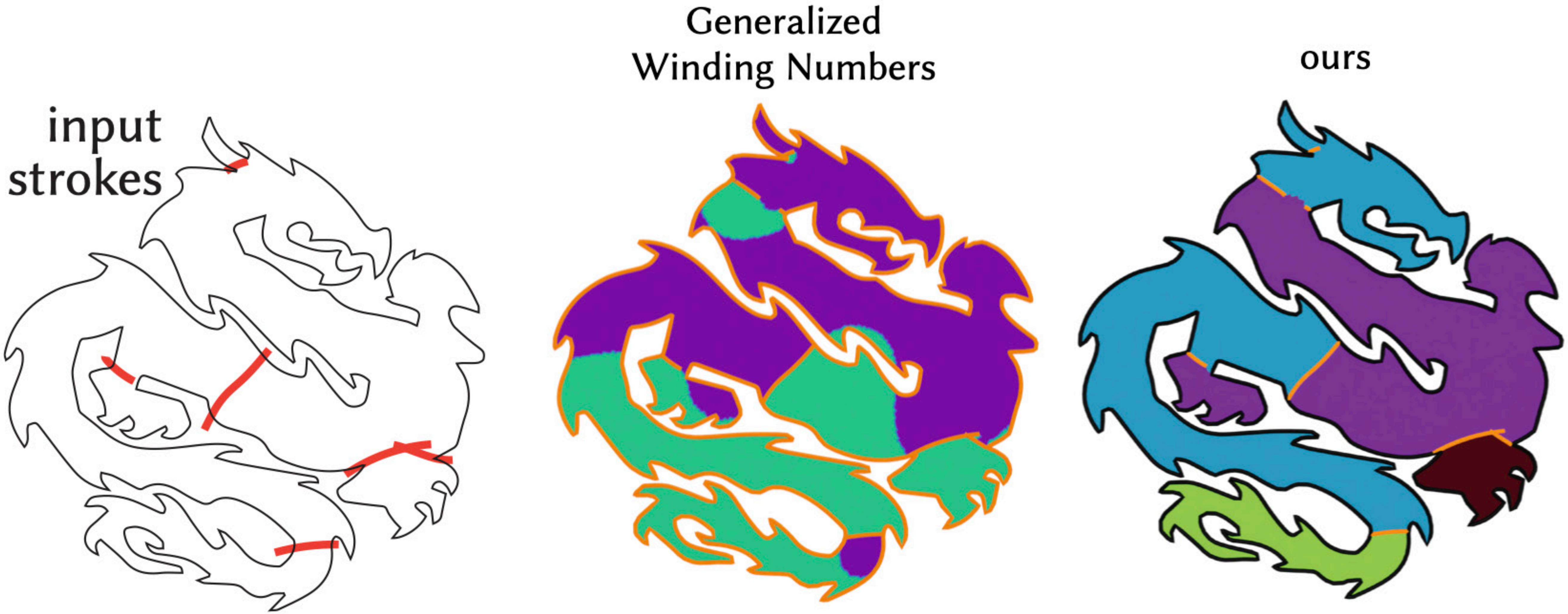

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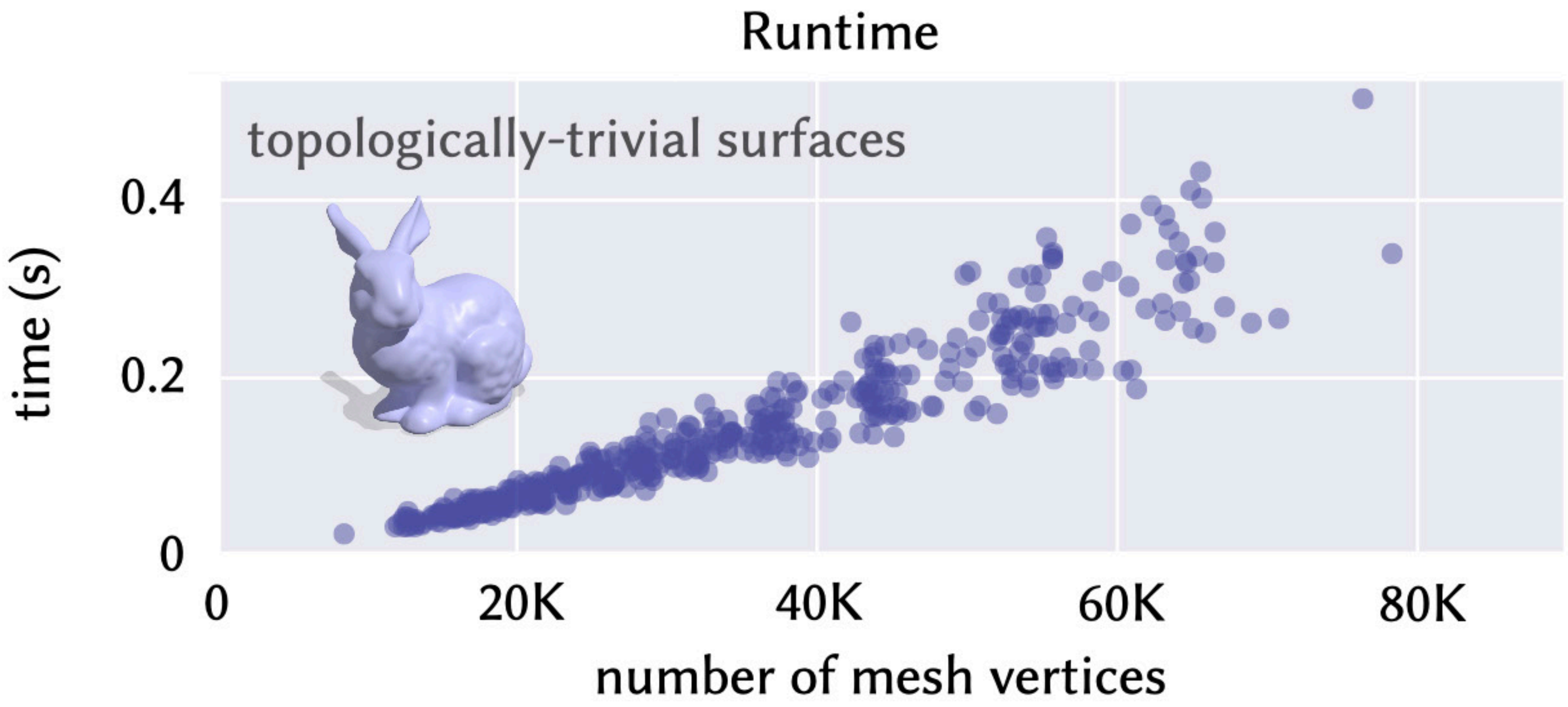
Performance



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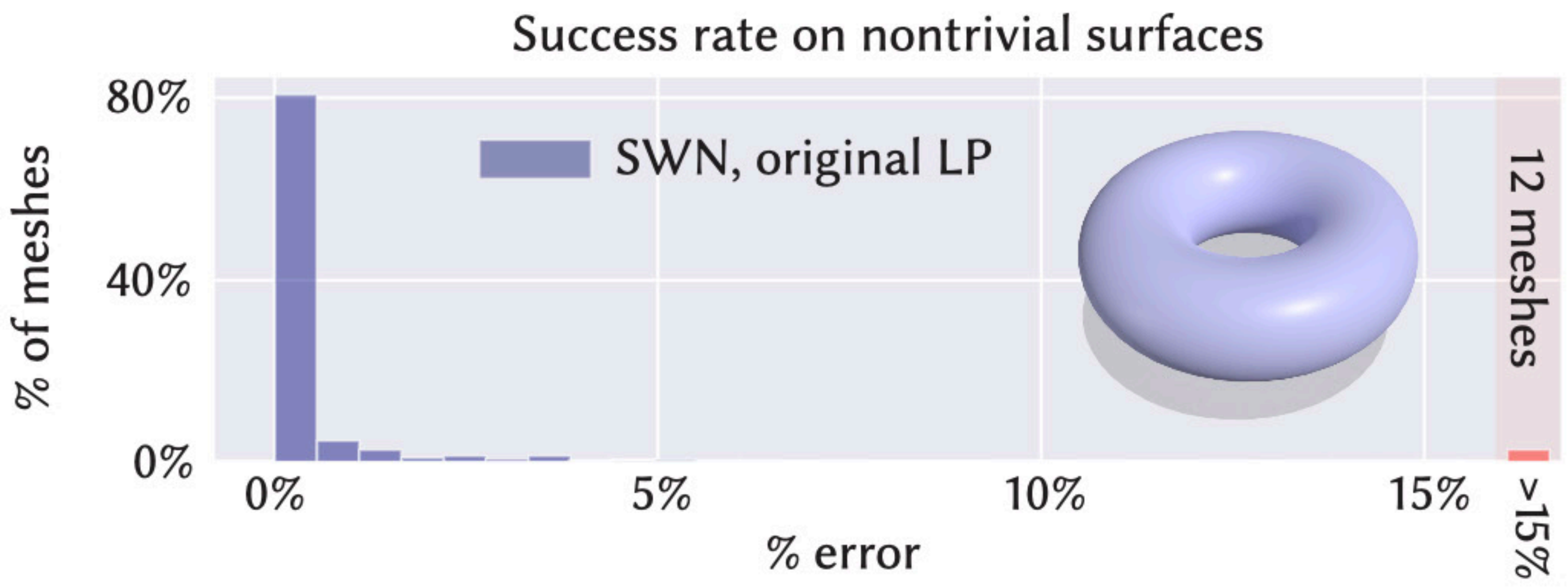

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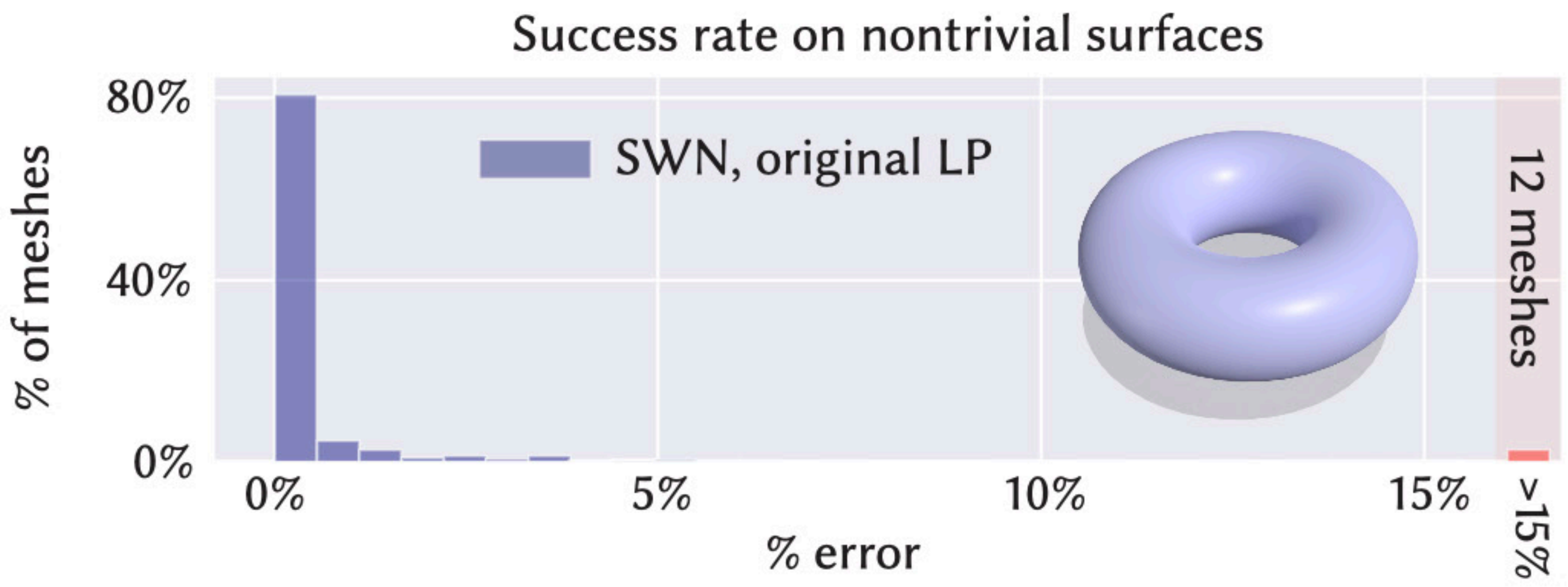

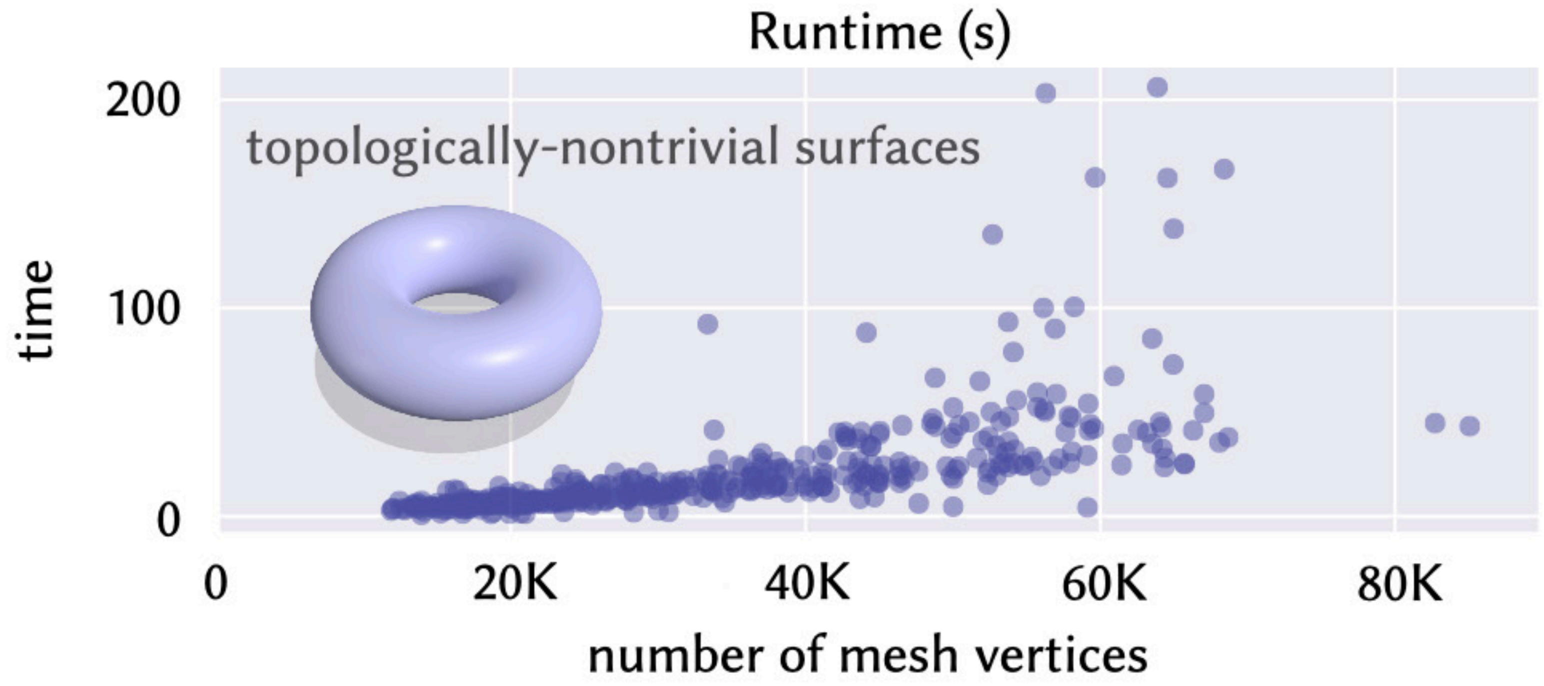

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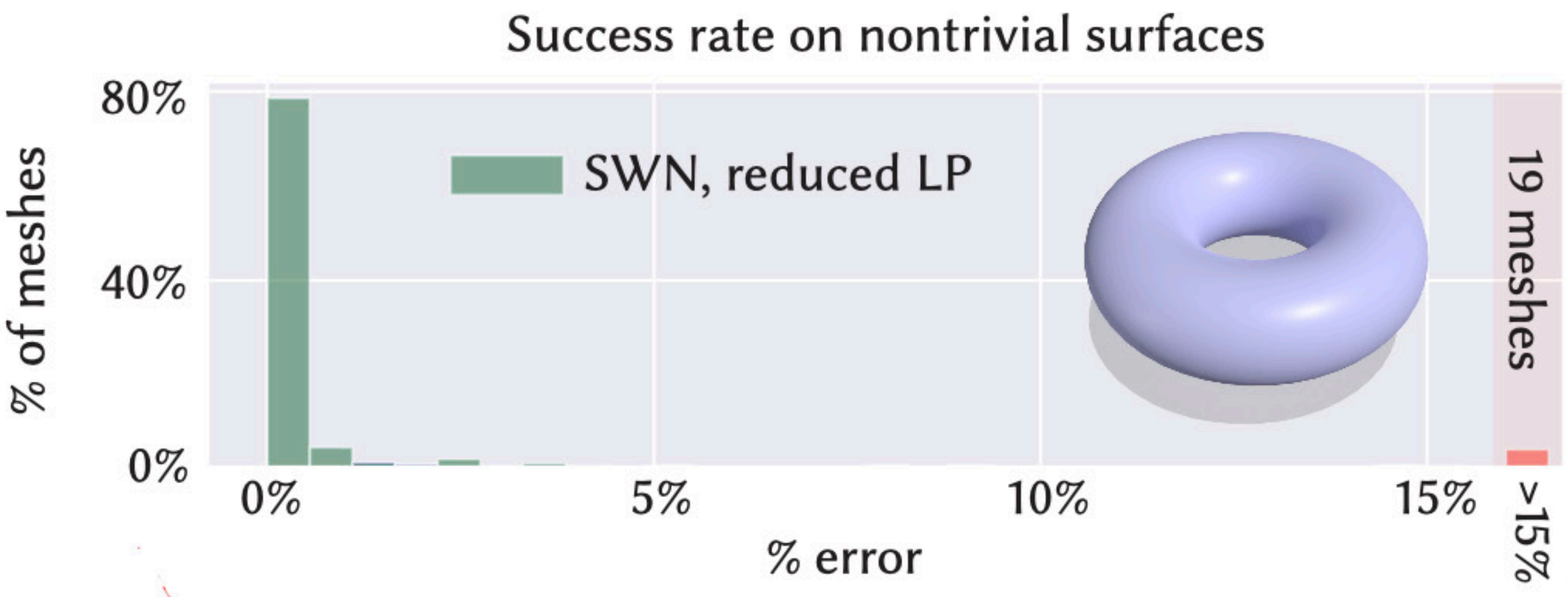

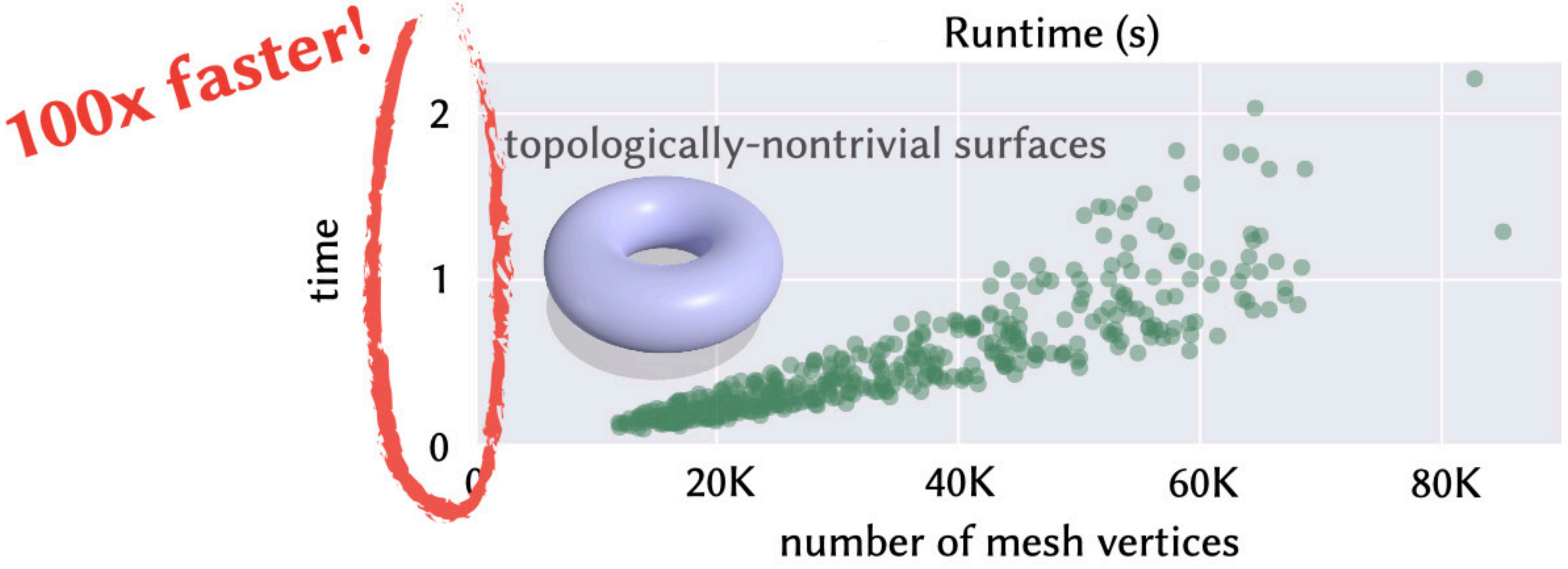

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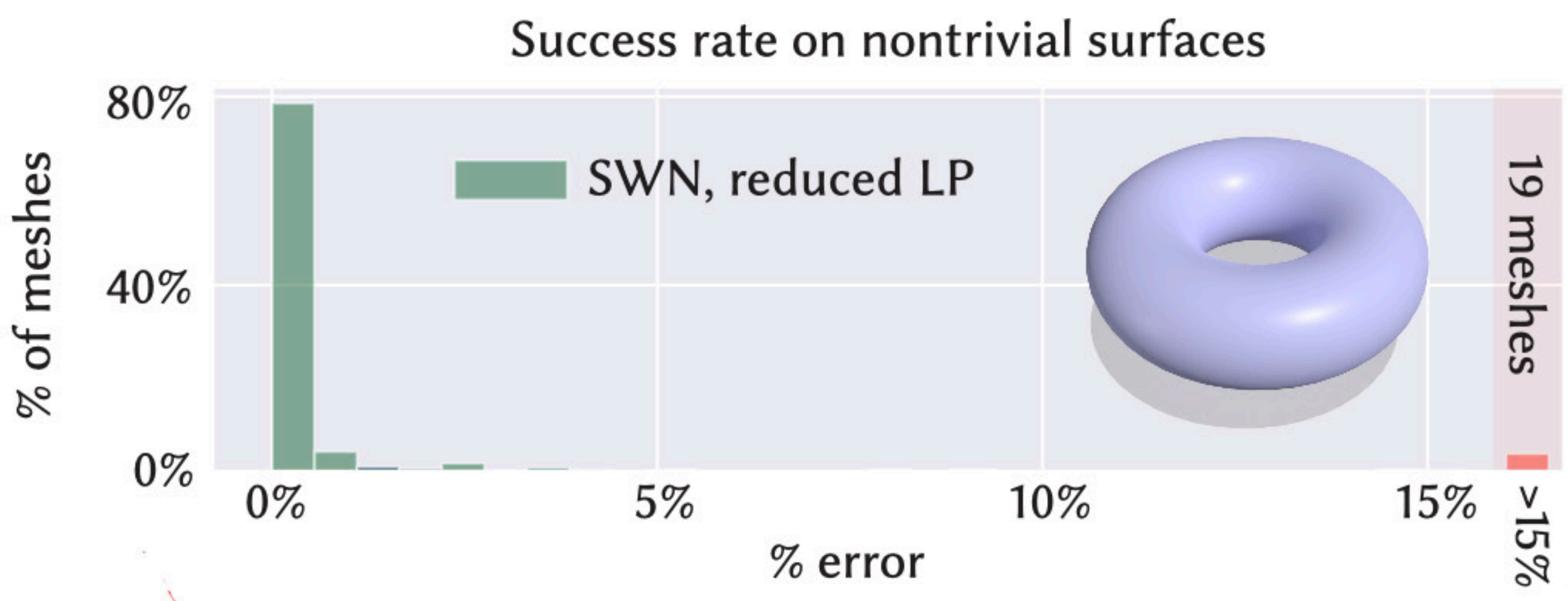

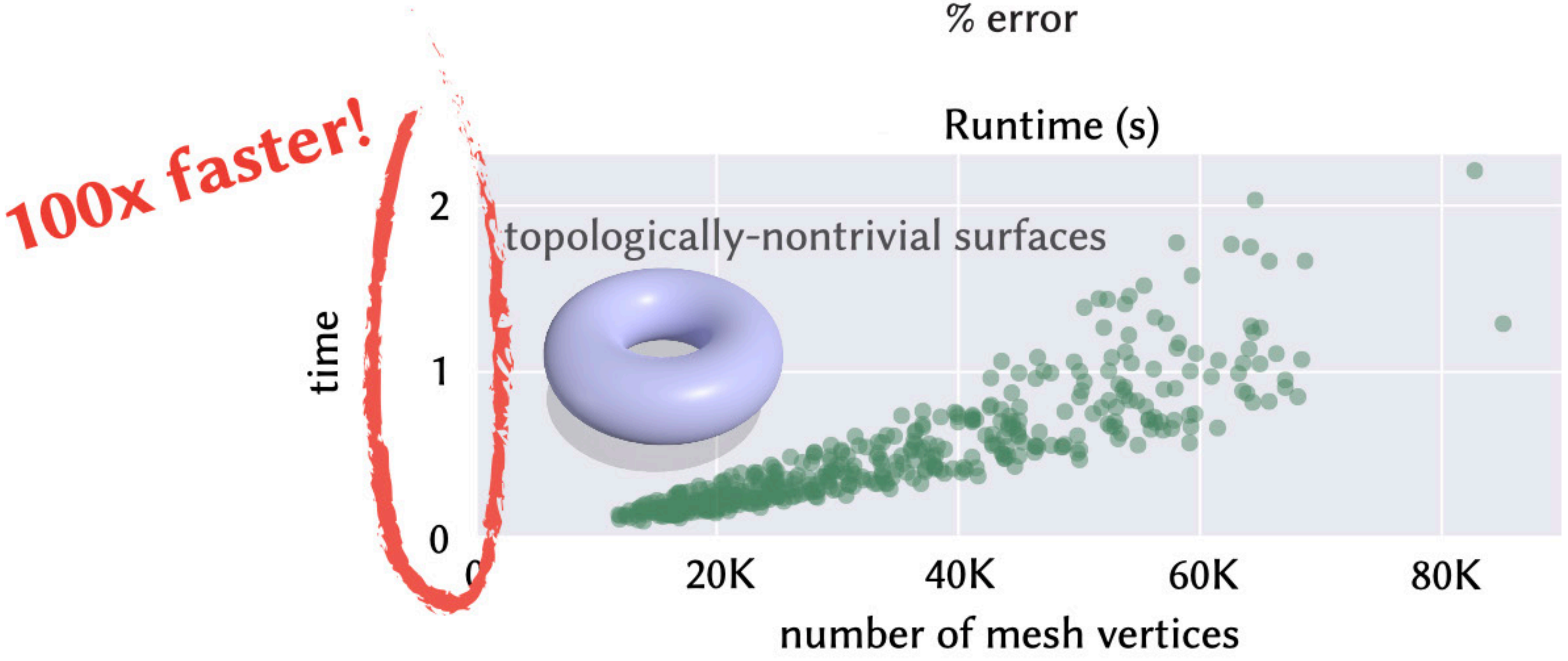

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Conclusion

Theory

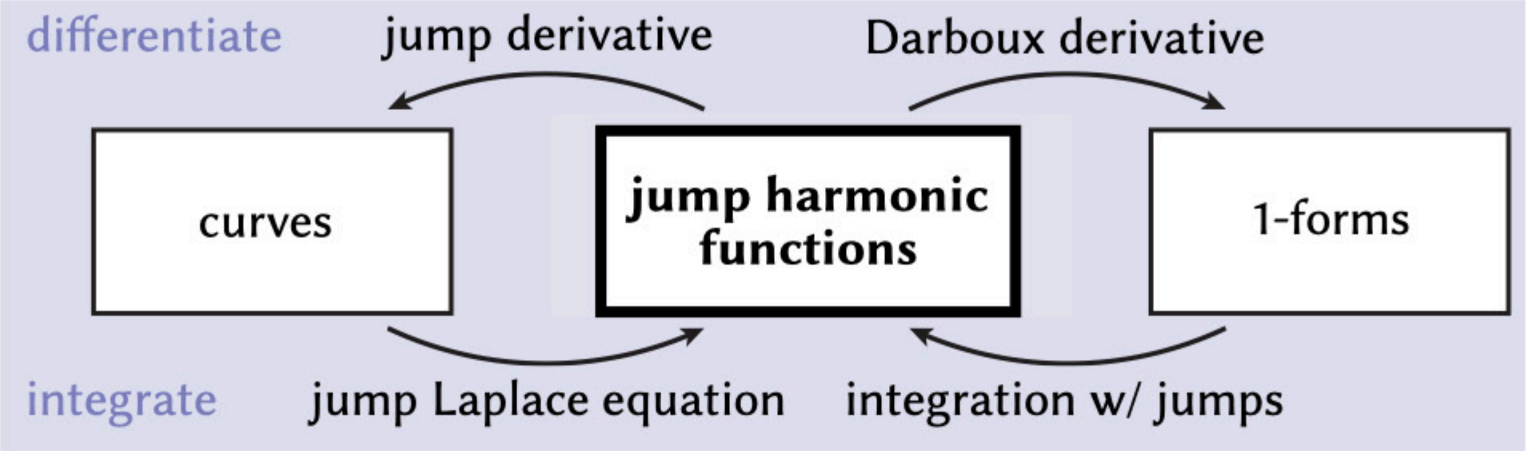
Theory

Cohomology → robust homological geometry processing

Theory

Cohomology → robust homological geometry processing

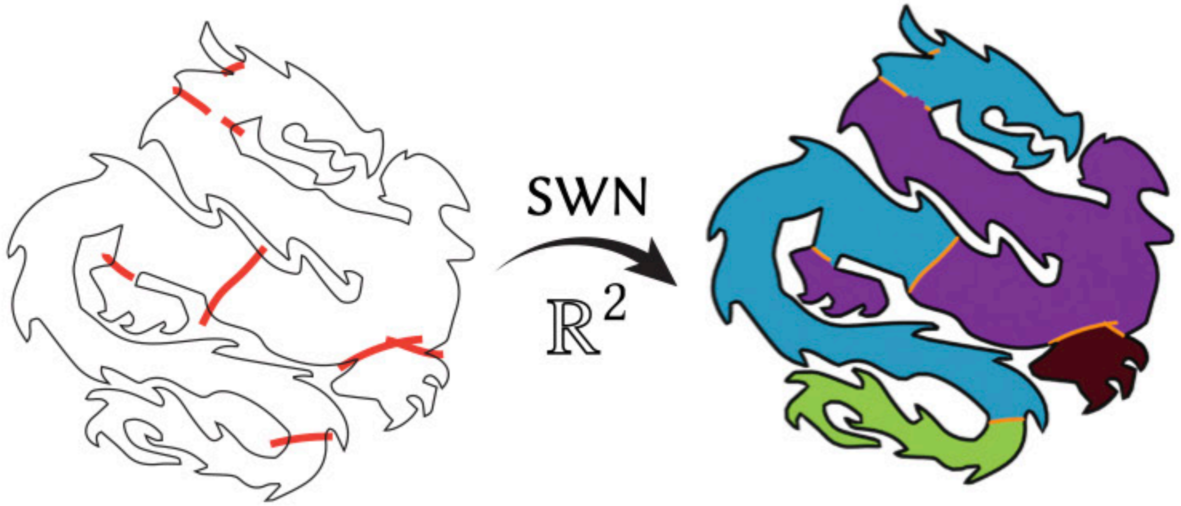
Duality between curves and 1-forms → jump harmonic functions to translate between the two



Fun future directions

Fun future directions

Subsets of $\mathbb{R}^n$:



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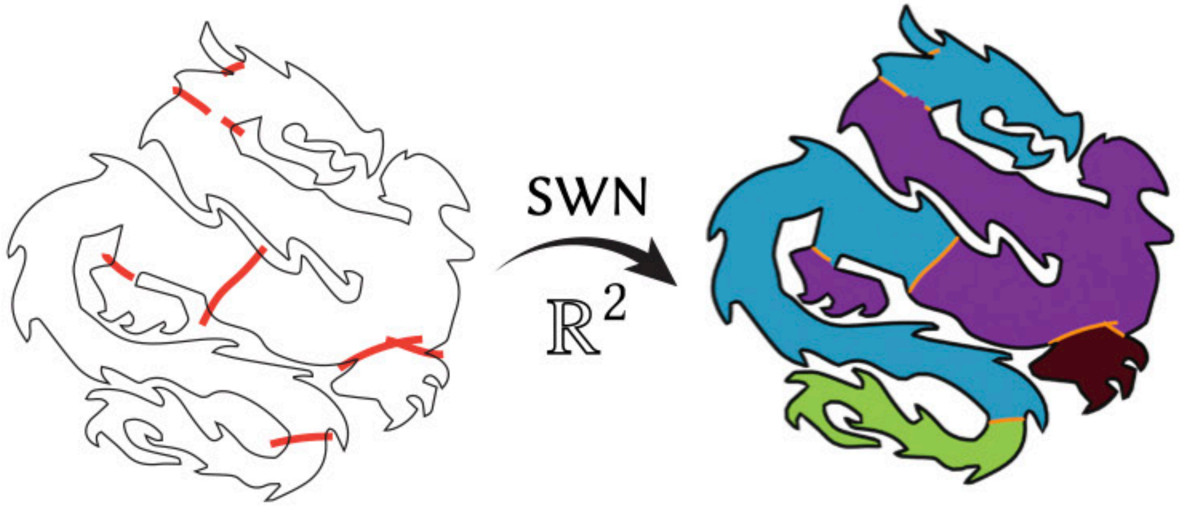

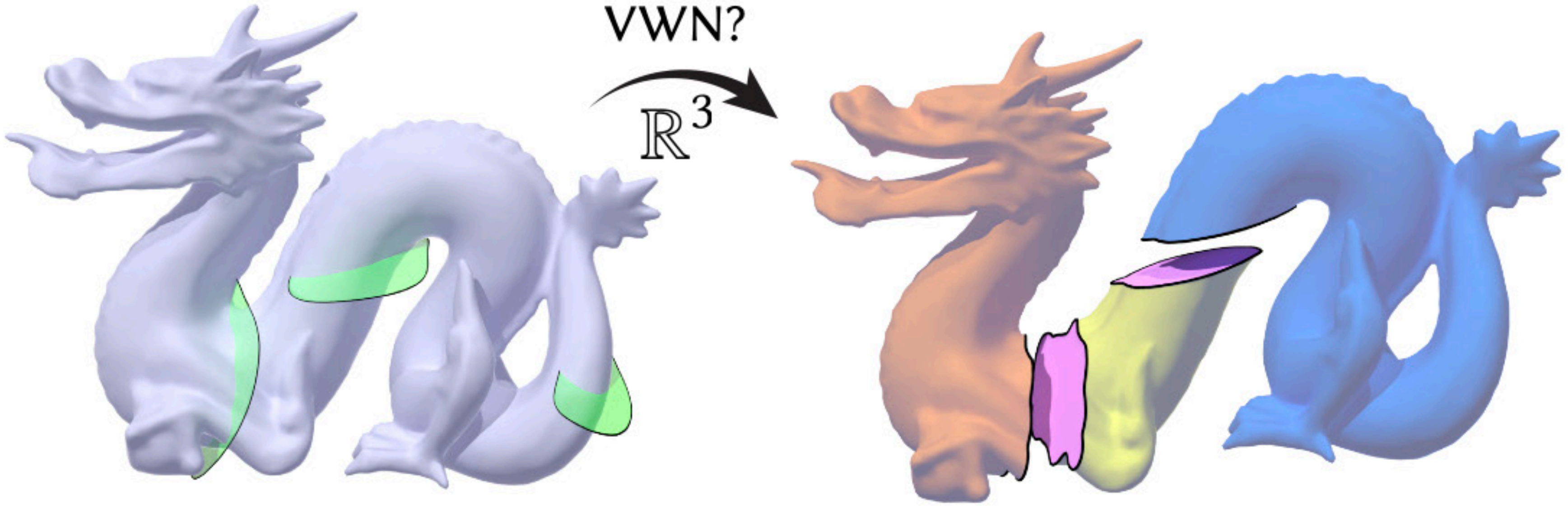

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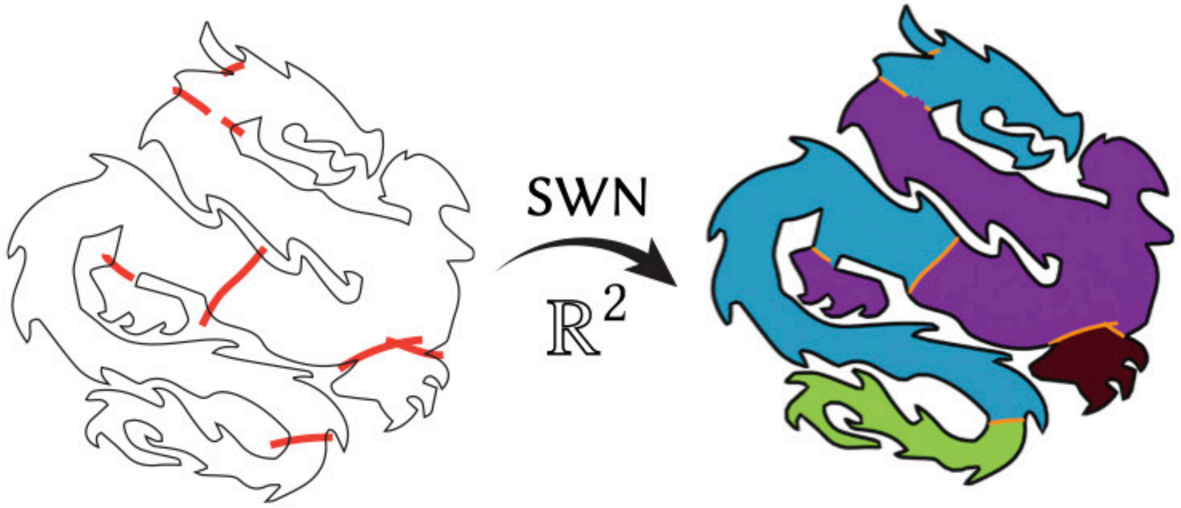
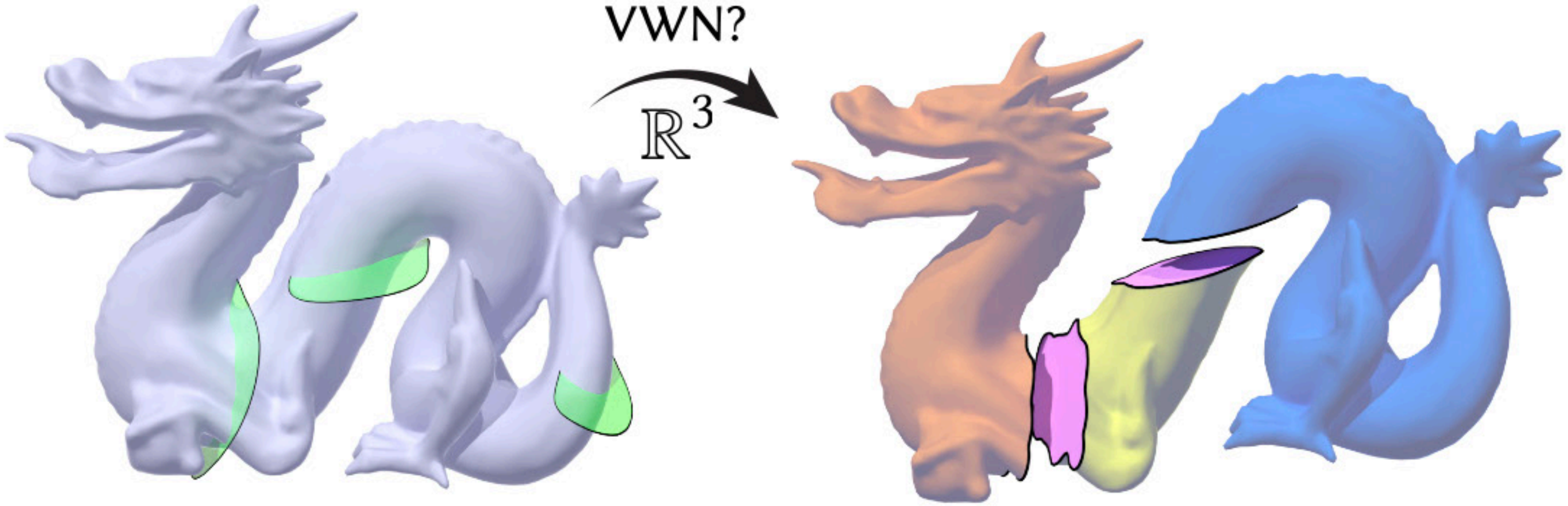
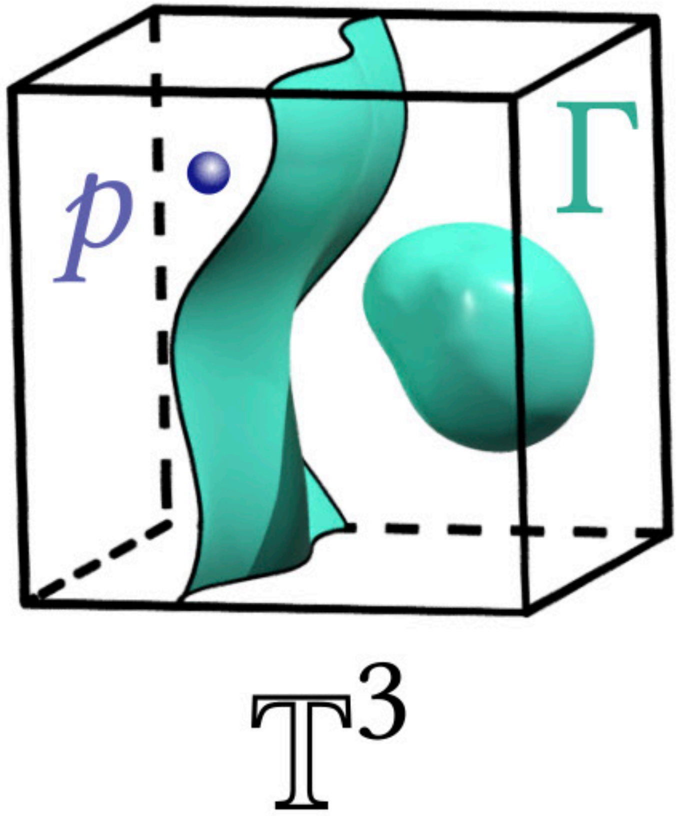




Extension of SWN to higher dimensions, e.g. periodic domains in 3D.



THANKS!